Rational CFTs on Riemann surfaces

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Abstract

The partition function of rational conformal field theories (CFTs) on Riemann surfaces is expected to satisfy ODEs of Gauss-Manin type. We investigate the case of hyperelliptic surfaces and derive the ODE system for the $(2, 5)$ minimal model.

Contents

1 Introduction

The present paper gives an ab initio mathematical introduction to rational conformal field theories (RCFT) on arbitrary genus $g \geq 1$ Riemann surfaces. Our approach requires only three relatively simple and neat axioms. The central objects are holomorphic fields and their N-point functions $\langle \phi_1 \dots \phi_N \rangle$. In order to actually compute these functions and more specifically the partition function $\langle \mathbf{1} \rangle$ for $N = 0$, one has to study their behaviour under changes of the conformal structure. This is done conveniently by first considering arbitrary changes of the metric. Such a change of $\langle \phi_1 \dots \phi_N \rangle$ is described by the corresponding $(N + 1)$-point function containing a copy of the Virasosoro field T. For this reason we have previously investigated the N-point functions of T (rather than of more general fields) [10]. In the present paper we study functions on the *moduli space* $\mathcal{M}_g$, which is the space of all possible conformal structures on the genus g surface. For the RCFTs one obtains functions which are meromorphic on a compactification of $\mathcal{M}_g$ or of a finite cover. We shall use that conformal structures occur as equivalence classes of metrics, with equivalent metrics being related by Weyl transformations. The N-point functions of a CFT do depend on the Weyl transformation, but only in a way which can be described by a universal automorphy factor.

For $g = 1$ this has been made explicit in [12]. Less is known about automorphic functions for $g > 1$. Our work develops methods in this direction. The basic idea is that many of the relevant functions are algebraic. In order to proceed step by step, we will restrict our investigations to the locus of hyperelliptic curves, though the methods work in more general context as well.

For an important class of CFTs (the minimal models), the zero-point functions $\langle \mathbf{1} \rangle$ will turn out to solve a linear differential equation so that $\langle \mathbf{1} \rangle$ can be computed for arbitrary hyperelliptic Riemann surfaces. Since $\langle \mathbf{1} \rangle$ is algebraic (namely a meromorphic function on a finite covering of the moduli space), it is clear a priori that the equation can not be solved numerically only, but actually analytically.

2 Notations and conventions

In this paper, $0^0 = 1$.

For any category $\mathfrak{Cat}$, we denote by $|\mathfrak{Cat}|$ the set of objects of $\mathfrak{Cat}$. For any pair of objects $O_1, O_2 \in |\mathfrak{Cat}|$, we denote by $\mathrm{Mor}_{\mathfrak{Cat}}(O_1, O_2)$ the set of morphisms $O_1 \to O_2$ of $\mathfrak{Cat}$.

Let $\mathfrak{Diff}$ be the category of differentiable manifolds, and of smooth maps. Here by smooth we mean C^∞.

By a *Riemann surface* we mean a one-dimensional complex manifold. If $U \subseteq \mathbb{C}$ is an open subset, we say that a map $f : U \to \mathbb{C}$ is *conformal* if f is biholomorphic on its image. Let $\mathfrak{Riem}$ be the category of (not necessarily compact) Riemann surfaces without boundary, and with conformal maps.

By a *Riemannian manifold* we mean a real smooth manifold equipped with a Riemannian metric, i.e. a smooth positive section in the symmetric square of the cotangent bundle of the manifold.

Our surfaces are non-singular, i.e. they have no multiple ramification points.

We shall use the convention [19]

$$G_{2k}(z) = \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^{2k}} + \frac{1}{2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^{2k}}\,,$$

and define E_{2k} by $G_k(z) = \zeta(k)E_k(z)$ for $\zeta(k) = \sum_{n\geq 1}\frac{1}{n^k}$, so e.g.

$$\begin{aligned}
G_2(z) &= \frac{\pi^2}{6}E_2(z)\,,\\
G_4(z) &= \frac{\pi^4}{90}E_4(z)\,,\\
G_6(z) &= \frac{\pi^6}{945}E_6(z)\,.
\end{aligned}$$

Let $(q)_n := \prod_{k=1}^n (1-q^k)$ be the q-Pochhammer symbol. The *Dedekind η function* is

$$\eta(z) := q^{\frac{1}{24}}(q)_\infty = q^{\frac{1}{24}}\left(1 - q + q^2 + q^5 + q^7 + \ldots\right)\,, \qquad q = e^{2\pi i\, z}\,.$$

For $q = e^{2\pi i\tau}$, the theta functions $\vartheta_i(z,q) = \vartheta_i(z)$ at $z = 0$ are given by [1, which however uses the convention $q = e^{\pi i\tau}$],

$$\begin{aligned}
\vartheta_1(0) &= 0\\
\vartheta_2(0) &= 2q^{1/8}\sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} = 2q^{1/8}(1 + q + q^3 + q^6 + q^{10} + \ldots)\\
\vartheta_3(0) &= 1 + 2\sum_{n=1}^{\infty} q^{\frac{1}{2}n^2} = 1 + 2q^{\frac{1}{2}} + 2q^2 + 2q^{\frac{9}{2}} + \ldots\\
\vartheta_4(0) &= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{\frac{1}{2}n^2} = 1 - 2q^{\frac{1}{2}} + 2q^2 - 2q^{\frac{9}{2}} + \ldots
\end{aligned}$$

We have the Jacobi identity:

$$\vartheta_2^4 + \vartheta_4^4 = \vartheta_3^4\,. \tag{1}$$

3 Preliminaries

We specify what we mean by a smooth category and introduce F-bundle functors. Subsequently we remind the reader of the definition of primary fields and of N-point functions.

3.1 Categories with a differentiable structure

Let $\mathfrak{Diff}$ be the category of differentiable manifolds.

Definition 1. *A category* $\mathfrak{Cat}$ ***has a differentiable structure*** *if*

1. $\forall\, O_1, O_2 \in |\mathfrak{Cat}|$, $\forall\, \Sigma_1, \Sigma_2 \in |\mathfrak{Diff}|$ *and for any smooth map*

$$f : \Sigma_2 \;\rightarrow\; Mor_{\mathfrak{Cat}}(O_1, O_2)\,,$$

the composition

$$f \circ \varphi : \Sigma_1 \;\rightarrow\; Mor_{\mathfrak{Cat}}(O_1, O_2)$$

is smooth, $\forall\, \varphi \in Mor_{\mathfrak{Diff}}(\Sigma_1, \Sigma_2)$*;*

2. $\forall\, O_1, O_2, O_3 \in |\mathfrak{Cat}|$, $\forall\, \Sigma_1, \Sigma_2 \in |\mathfrak{Diff}|$ *and for any pair of smooth maps*

$$f_i : \Sigma_i \;\rightarrow\; Mor_{\mathfrak{Cat}}(O_i, O_{i+1})\,, \quad i = 1, 2,$$

the induced map

$$\Sigma_1 \times \Sigma_2 \;\rightarrow\; Mor_{\mathfrak{Cat}}(O_1, O_3)$$

defined by $(z_1, z_2) \;\mapsto\; f_2(z_2) \circ f_1(z_1)$ *is smooth.*

Definition 2. *Let* $\mathfrak{Cat}$ *be a category with a differentiable structure. A functor* $F : \mathfrak{Cat} \rightarrow F(\mathfrak{Cat})$ *is* ***smooth*** *if*

1. $F(\mathfrak{Cat})$ *has a differentiable structure,*

2. $\forall\, O_1, O_2 \in |\mathfrak{Cat}|$,

$$Mor_{\mathfrak{Cat}}(O_1, O_2) \;\rightarrow\; Mor_{F(\mathfrak{Cat})}(F(O_1), F(O_2))$$

is smooth.

3.2 F-bundle functors

Let F be an infinite dimensional $\mathbb{C}$-vector space endowed with an ascending filtration by finite-dimensional subvector spaces

$$F_0 \subset F_1 \subset F_2 \subset \ldots\,, \quad F = \cup_{i \in \mathbb{N}_0} F_i\,.$$

Equip F with the finest topology for which the inclusions $F_i \subset F$ for $i \geq 0$ are continuous. Equivalently, a series $(\mathbf{x}^i)_{i \in \mathbb{N}}$ with $\mathbf{x}^i \in F$ for $i \in \mathbb{N}$ converges to $\mathbf{x} \in F$ iff

1. $\exists\, m_0 \in \mathbb{N}$ such that $\mathbf{x} \in F_{m_0}$ and $\mathbf{x}^i \in F_{m_0}$, $\forall\, i \in \mathbb{N}$,

2. $(\mathbf{x}^i)_{i \in \mathbb{N}}$ converges to $\mathbf{x}$ in F_{m_0}.

Definition 3. *We refer to* F *as a* ***quasi-finite*** $\mathbb{C}$***-vector space****.*

The filtration induces a grading

$$F = \oplus_{i\in\mathbb{N}_0} F_i/F_{i-1}$$

of F into finite-dimensional complex subvector spaces. Let

$$\mathrm{End}_{\mathrm{grad}}(F) \cong \oplus_i \mathrm{End}(F_i/F_{i-1})$$

be the ring of endomorphisms of F that respect this grading. These are the only endomorphisms of F we will consider. In a basis of F, an element $A \in \mathrm{End}_{\mathrm{grad}}(F)$ can be written as a block diagonal matrix in which the i'th block defines an element $A_i \in \mathrm{End}(F_i/F_{i-1})$. A is smooth (we mean C^∞) if for every $i \in \mathbb{N}_0$, A_i is smooth on the real vector space underlying F_i/F_{i-1}.

Definition 4. *Let F be a quasi-finite $\mathbb{C}$-vector space.*

1. *By a **vector bundle $\mathcal{E}$ with fiber** F we mean a family of pairs $(\mathcal{E}_i, \iota_i)$ for $i \in \mathbb{N}_0$, where $\mathcal{E}_i$ is a vector bundle with standard fiber F_i, and $\iota_i : \mathcal{E}_i \subset \mathcal{E}_{i+1}$ is an inclusion of vector bundles.*

2. *For any two vector bundles $\mathcal{E}, \mathcal{E}'$ with fiber F, a **morphism** $f : \mathcal{E} \to \mathcal{E}'$ **of vector bundles with fiber** F is a family of vector bundle morphisms $f_i : \mathcal{E}_i \to \mathcal{E}'_i$ with*

$$f_{i+1}|_{\mathcal{E}_i} = f_i$$

for $i \in \mathbb{N}_0$, where $\mathcal{E}_i$ and $\mathcal{E}'_i$ are the vector bundles with standard fiber F_i defined by $\mathcal{E}$ and $\mathcal{E}'$, respectively.

3. *In particular, if $\mathcal{E}$ and $\mathcal{E}'$, respectively, is a vector bundle over a smooth manifold, then f is **smooth** if f_i is smooth for every $i \in \mathbb{N}_0$.*

*We define $\mathfrak{Vec}(F)$ to be the category of vector bundles with fiber F, and with smooth morphisms. The objects in $|\mathfrak{Vec}(F)|$ are referred to as **F-bundles**.*

The morphism set of the category $\mathfrak{Riem}$ and $\mathfrak{Vec}(F)$, respectively, has a natural manifold structure:

Propos. 5. *For $\Sigma, \Sigma' \in |\mathfrak{Riem}|$, the set $Mor_{\mathfrak{Riem}}(\Sigma, \Sigma')$ is naturally an infinite dimensional complex manifold. For $\mathcal{E}, \mathcal{E}' \in |\mathfrak{Vec}(F)|$, we have $Mor_{\mathfrak{Vec}(F)}(\mathcal{E}, \mathcal{E}') \in |\mathfrak{Diff}|$ in a natural way.*

Proof. Let $\Sigma, \Sigma' \in |\mathfrak{Riem}|$, and let M be any complex manifold. We say that $\varphi : M \to \mathrm{Mor}_{\mathfrak{Riem}}(\Sigma, \Sigma')$ is *holomorphic* if the induced map $\varphi_1 : M \times \Sigma \to \Sigma'$ defined by $\varphi_1(p, q) := (\varphi(p))(q)$ for $p \in M$, $q \in \Sigma$ is holomorphic. The proof of the statement for $\mathfrak{Vec}(F)$ is analogous. □

In the following, we shall treat $\mathrm{Mor}_{\mathfrak{Riem}}(\Sigma_1, \Sigma_2)$ as a smooth manifold (by forgetting about its complex structure).

Definition 6. *For any quasi-finite $\mathbb{C}$-vector space F, an **F-bundle functor** is a covariant functor*

$$\Phi_F : \mathfrak{Riem} \to \mathfrak{Vec}(F)$$

with the following properties:

- *$\forall\, \Sigma \in |\mathfrak{Riem}|$, $\Phi_F(\Sigma) =: \mathcal{F}_\Sigma$ is a vector bundle over Σ,*

- Φ_F *is compatible with restrictions: if* $U \subset \Sigma$ *then* $\mathcal{F}_U = \mathcal{F}_\Sigma|_U$,
- $\forall\, \Sigma_1, \Sigma_2 \in |\mathfrak{Riem}|$, Φ_F *defines an element in*

$$Mor_{\mathfrak{Diff}}(Mor_{\mathfrak{Riem}}(\Sigma_1, \Sigma_2), Mor_{\mathfrak{Vec}(F)}(\mathcal{F}_{\Sigma_1}, \mathcal{F}_{\Sigma_2}))\ .$$

Example 7. *The tangent functor* $T : \mathfrak{Diff} \to \mathfrak{Diff}$ *has precisely the above listed properties: For* $M \in |\mathfrak{Diff}|$, TM *defines the the tangent bundle over* M, *and if* $f \in Mor_{\mathfrak{Diff}}(M, N)$, *we have* $Tf = df \in Mor_{\mathfrak{Diff}}(TM, TN)$. *Moreover, if* (U, z) *is a chart on* M, $Tz = dz$ *defines a nowhere vanishing section in the cotangent bundle* T^*U, *and thus a trivialisation* $TU \cong U \times \mathbb{C}$.

The latter observation is actually a general feature.

Propos. 8. Φ_F *defines a canonical trivialisation of* $\mathcal{F}_\mathbb{C} = \Phi_F(\mathbb{C})$ *with fiber* $\mathcal{F}_{\mathbb{C},0} = F$.

Proof. All conformal self-maps of $\mathbb{C}$ are affine linear. For $z \in \mathbb{C}$, let $t_z : \mathbb{C} \to \mathbb{C}$ be the translation by z. The induced morphism $\Phi_F(t_z)$ maps $F = \mathcal{F}_{\mathbb{C},0}$ isomorphically to $\mathcal{F}_{\mathbb{C},z}$. The map $\mathbb{C} \times F \to \mathcal{F}_\mathbb{C}$ defined by $(z, \varphi) \mapsto (\Phi_F(t_z))(\varphi) \in \mathcal{F}_{\mathbb{C},z}$ is invertible. □

If $U \in |\mathfrak{Riem}|$ has coordinate $z : U \to \mathbb{C}$, $\Phi_F(U)$ trivialises in a way determined by $\Phi_F(z)$. For $(p, \varphi) \in \mathbb{C} \times F$, the corresponding element in $\mathcal{F}_U$ is

$$\varphi_z(p) = (\Phi_F(z))^{-1}(p, \varphi)\ .$$

Abusing notations, we shall simply write $\varphi(z)$ where we actually mean $\varphi_z(p)$. (This will entail notations like $\hat{\varphi}(\hat{z})$ instead of $\varphi_{\hat{z}}(p)$ etc.)

We shall only consider bundles that lie in $|\mathfrak{Vec}(F)|$.

3.3 Primary fields

Let $O_\mathbb{C}$ be the sheaf of germs $\langle U, f\rangle$ which are represented by pairs (U, f) for some open set $U \subseteq \mathbb{C}$ and some conformal map $f : U \to \mathbb{C}$. Let $O_{\mathbb{C},0}$ be the fiber of germs in $O_\mathbb{C}$ which are defined at the origin in $\mathbb{C}$, and let

$$G := \{\langle U, f\rangle \in O_{\mathbb{C},0} \mid f(0) = 0\}\ .$$

It is easy to see that G is a group under pointwise composition, with identity element $\langle \mathbb{C}, \mathrm{id}\rangle$. G is actually a Lie group [15, p. 267]. G is a real manifold that admits no complexification.

The Lie algebra $\mathfrak{g}$ of G can be identified with the Lie algebra of germs of holomorphic vector fields on $\mathbb{C}$ which vanish at the origin [3],

$$\mathfrak{g} = \mathrm{span}_\mathbb{R}\{\langle U, \ell_n\rangle\}_{n\geq 0}\ ,$$

where $\ell_n = -z^{n+1}\partial_z$. These polynomial vector fields define diffeomorphisms of S^1 that extend to the unit disc $\{z \in \mathbb{C}||z| \leq 1\}$. Over $\mathbb{C}$, the vector fields ℓ_n for $n \in \mathbb{Z}$ generate the Witt algebra. [17, p. 34]. The infinite-dimensional Lie group Diff(S^1) of orientation preserving diffeomorphisms of S^1 has no complexification.

Propos. 9. Φ_F *defines a representation of* G *on* F.

Proof. For any pair of representatives (U, f) and (V, g) of a germ $\langle U, f\rangle \in G$, the corresponding bundle maps $\Phi_F(f)$ and $\Phi_F(g)$ induce the same automorphism of $F = \mathcal{F}_{\mathbb{C},0}$. □

By assumption, the representation decomposes into finite-dimensional subrepresentations, corresponding to the grading of F. The corresponding representation of the Lie algebra

$$\mathfrak{g} \;\to\; \mathrm{End}_{\mathrm{grad}}(F)$$

extends to an $\mathbb{R}$-linear representation $L + \bar{L}$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$, where L and $\bar{L}$ are complex linear and a complex antilinear Lie algebra homomorphisms, respectively. For $n \geq 0$, let L_n and $\bar{L}_n$ be the image of $\langle U, \ell_n \rangle$ under L and $\bar{L}$, respectively, in $\mathrm{End}_{\mathrm{grad}}(F)$. $\{L_n\}_{n\geq 0}$ satisfy a Lie subalgebra of the Witt algebra

$$[L_n, L_m] = (n-m)L_{n+m}\,. \tag{2}$$

$\{\bar{L}_n\}_{n\geq 0}$ define an isomorphic Lie algebra, and $[\bar{L}_n, L_m] = 0$ for $n, m \geq 0$.

For $n \geq 0$, $L_n + \bar{L}_n$ and $i(L_n - \bar{L}_n)$ represent the generator of the infinitesimal transformations

$$\begin{aligned} z &\mapsto \exp(-\varepsilon z^{n+1}\partial_z) \approx z(1-\varepsilon z^n)\,, \\ z &\mapsto \exp(-i\varepsilon z^{n+1}\partial_z) \approx z(1-i\varepsilon z^n)\,, \quad \varepsilon > 0\,, \quad z \in \mathbb{C}\,, \end{aligned}$$

respectively. ($\bar{z}$ is treated as an independent variable and will be disregarded.) In particular, $L_0 + \bar{L}_0$ and $i(L_0 - \bar{L}_0)$ represent the generator of the infinitesimal dilation and rotation, respectively, in a one-dimensional complex vector space.

Propos. 10. *Let V be a complex representation of G, $\dim_{\mathbb{C}} V = 1$, such that*

$$L_0|_V = h \cdot \mathrm{id}_V\,, \quad \bar{L}_0|_V = \bar{h} \cdot \mathrm{id}_V\,, \tag{3}$$

for some pair of numbers $h, \bar{h} \in \mathbb{R}$. Then $h - \bar{h} \in \mathbb{Z}$, and

$$L_n|_V = 0 \quad \text{for } n > 0\,. \tag{4}$$

Proof. By eq. (3), $L_0 - \bar{L}_0 = h - \bar{h}$ in V. Now $\exp(i\varepsilon(L_0 - \bar{L}_0))$ defines a rotation by ε in V, so taking $\varepsilon = 2\pi$ shows that $h - \bar{h} \in \mathbb{Z}$. Now let $V = \mathrm{span}_{\mathbb{C}}\{v\}$ for some simultaneous eigenvector $v \neq 0$ of L_0 and $\bar{L}_0$. Since $[L_n, L_0] \neq 0$ for $n > 0$, we have $L_n v = 0$ in this case. □

Definition 11. *An element $\varphi \in F$ has the property of being **primary** if $\mathrm{span}_{\mathbb{C}}\langle\varphi\rangle$ defines a one-dimensional representation of G.*

We give a converse to Proposition 9.

Propos. 12. *F-bundle functors Φ_F are characterised, up to bundle isomorphisms, by representations of G.*

Proof. Let V be a complex one-dimensional representation of G with property (3). Suppose $V = \mathrm{span}_{\mathbb{C}}\{v\}$ for some vector $v \in F$. By definition

$$v \in F_n \tag{5}$$

if $h - \bar{h} \leq n$. This defines a grading $F = \oplus_{i\in\mathbb{N}_0} F_i/F_{i-1}$. Since Φ_F is compatible with restrictions, it suffices to define the functor locally. For two contractible sets $U, V \in |\mathfrak{Riem}|$ and for $f \in \mathrm{Mor}_{\mathfrak{Riem}}(U, V)$, $\Phi_F(U) \cong U \times F$, so $\Phi_F(f)$ is determined by f and the representations $F_i/F_{i-1} \to F_i/F_{i-1}$ of G, for $i \in \mathbb{N}_0$. □

We shall come back to our standard example and consider tensorial powers of tangent line bundle $T\mathbb{C}$ and its complex conjugate $\overline{T\mathbb{C}}$.

Propos. 13. *Every rank-one subbundle of $\Phi_F(\mathbb{C})$ is isomorphic to a bundle of the form*

$$(T\mathbb{C})^{h-\bar{h}} \otimes (T\mathbb{C} \otimes \overline{T\mathbb{C}})^{\bar{h}}$$

*with $\bar{h} \in \mathbb{R}_0^+$ and $h - \bar{h} \in \mathbb{Z}$. We refer to this bundle as the $(h, \bar{h})$-**bundle** and write*

$$(T\mathbb{C})^h \otimes (\overline{T\mathbb{C}})^{\bar{h}} \ .$$

Note that the latter should be taken as a notation only. Since under coordinate change $z \mapsto w$, $T\mathbb{C}$ has the holomorphic transition function $\frac{dw}{dz}$, the transition function of $T\mathbb{C} \otimes \overline{T\mathbb{C}}$ is $\left|\frac{dw}{dz}\right|^2$, which is real and positive. Thus it has a well-defined logarithm.

Proof. Example 7 shows that $T\mathbb{C}$ and thus every $(h, \bar{h})$-bundle defines a rank-one subbundle of an F-bundle. To prove the converse, it suffices by Proposition 12 to show that every $(h, \bar{h})$-bundle, or in fact the differential functor T defines a one-dimensional representation of G which isomorphic to (3) and (4).

For infinitesimal $\varepsilon > 0$ and for $n \geq 0$, define $F_n : \mathbb{C} \to \mathbb{C}$ by $F_n(z) = z(1 + \varepsilon z^n)$. F_n defines an element in G. Since

$$T_0(F_m \circ F_n) = d(F_m \circ F_n)_0 = (F'_m \circ F_n)(0)F'_n(0)\, dz_0 = F'_m(0)F'_n(0)\, dz_0 \ ,$$

T defines a one-dimensional representation of G by

$$F_n \mapsto F'_n(0) \ .$$

Since $F'_n(0) = \exp(\varepsilon\delta_{n,0})$, the representation is isomorphic to that generated by L_n for $n \geq 0$, in V. We have a similar description for $\overline{T}$ and anti-holomorphic functions, and obtain a representation isomorphic to that generated by $\bar{L}_n$ for $n \geq 0$. □

3.4 States and N-point functions

For $N \geq 0$, define

$\mathcal{M}_{g,N}$ the moduli space of compact Riemann surfaces Σ of genus g with N different distinguished points $p_1, \ldots, p_N \in \Sigma$;

$\mathcal{M}^F_{g,N}$ the moduli space of Riemann surfaces $\Sigma \in \mathcal{M}_{g,N}$, on which for $1 \leq i \leq N$, there is a copy of F attached to p_i with one marked point $\varphi_i \in F$.

Let $o_N : \mathcal{M}^F_{g,N} \to \mathcal{M}_{g,0}$ be the forgetful map for $N > 0$ and the identity otherwise. Conversely, from $\Sigma \in \mathcal{M}_{g,0}$ we recover an element in $\mathcal{M}^F_{g,1}$ $(N = 1)$ by choosing a point $p \in \Sigma$ and marking an element $\varphi(p)$ in the fiber $\mathcal{F}_{\Sigma,p}$ of $\mathcal{F}_\Sigma = \Phi_F(\Sigma)$. We may view $\mathcal{F}_\Sigma$ as the set of elements in $\mathcal{M}^F_{g,1}$ that corresponds to all possible markings,

$$\mathcal{F}_\Sigma \cong o_1^{-1}\Sigma \ .$$

This description allows to vary the markings $p \in \Sigma$ and $\varphi(p) \in \mathcal{F}_{\Sigma,p}$ in a continuous way.

We will also have to have to discuss Riemannian metrics which are compatible with a given complex structure. Let S be a compact oriented genus g surface with a differentiable structure, (determined up to diffeomorphism). Let $\mathrm{Met}_g(S)$ be the additive semi-group of Riemannian metrics G on S or equivalently, the set of Riemannian surfaces $\tilde{S}$ diffeomorphic to S. (We shall use the two descriptions interchangeably.) We have the well-known isomorphism [2]

$$\mathcal{M}_{g,0} \cong \mathrm{Met}_g(S)/\mathrm{Weyl} \ltimes \mathrm{Diffeo}\,.$$

The map $\tilde{o} : \mathrm{Met}_g(S) \to \mathcal{M}_{g,0}$ is given by forgetting about the specific Riemannian metric G on $\tilde{S} \in \mathrm{Met}_g(S)$ and keeping only its conformal class $[G]$.

Definition 14. *For any $N \geq 0$, we define*

$$M^F_{g,N} := \{(\tilde{S}, \Sigma) \in Met_g(S) \times \mathcal{M}^F_{g,N} |\, \tilde{o}\tilde{S} = o_N\Sigma \text{ in } \mathcal{M}_{g,0}\}\,.$$

*For $N = 0$, we write $M_{g,0}$. An **N-point function** is a map*

$$\langle\quad\rangle : \quad M^F_{g,N} \to \mathbb{C}$$

which is

- *continuous as a function of $\tilde{S} \in Met_g(S)$, or of the metric G on S,*
- *smooth as a function on $o_N\Sigma \in \mathcal{M}_{g,0}$, and N-linear on the fibers F of $\Sigma \in \mathcal{M}^F_{g,N}$.*

*A **state** is a family of N-point functions for $N \in \mathbb{N}_0$.*

Since the marked points $p_1, \ldots, p_N$ on the Riemann surface are all distinct, for the purpose of local variations we replace an element $\Sigma \in \mathcal{M}^F_{g,N}$ with markings at $(p_1, \varphi_1(p_1)), \ldots, (p_N, \varphi_N(p_N))$ with the N-fold symmetric fiber product of elements in $\mathcal{M}^F_{g,1}$ defined on $o_N\Sigma$, where for $1 \leq i \leq N$, the ith factor is marked at $(p_i, \varphi_i(p_i))$.

More specifically, suppose $\Sigma \in \mathcal{M}_{g,0}$. We restrict the N-fold Cartesian product $\mathrm{sym}^{\times N}(\Sigma)$ of Σ to the locus

$$\mathrm{sym}^{\times N}_{\mathrm{restr}}(\Sigma) := \mathrm{sym}^{\times N}(\Sigma) \setminus \{(z_1 \ldots, z_N)|\, z_i = z_j \text{ for some } i \neq j\}$$

off partial diagonals. Moreover, let $\mathcal{F}_\Sigma = \Phi_F(\Sigma)$ with fiber $\mathcal{F}_{\Sigma,p}$ at $p \in \Sigma$, and let $\mathrm{sym}^{\boxtimes N}(\mathcal{F}_\Sigma)$ be its N-fold symmetric fiber product. We define $\mathrm{sym}^{\boxtimes N}_{\mathrm{restr}}(\mathcal{F}_\Sigma)$ to be the set obtained by restricting $\mathrm{sym}^{\boxtimes N}(\mathcal{F}_\Sigma)$ to the set of tensor products $\mathcal{F}_{\Sigma,p_1} \otimes \ldots \otimes \mathcal{F}_{\Sigma,p_N}$ with $(p_1, \ldots, p_N) \in \mathrm{sym}^{\times N}_{\mathrm{restr}}(\Sigma)$. Thus

$$\mathrm{sym}^{\boxtimes N}_{\mathrm{restr}}(\mathcal{F}_\Sigma) \cong o_N^{-1}\Sigma\,.$$

To conclude, let $(G, \Sigma) \in M_{g,0}$ and let

$$P : \mathrm{sym}^{\boxtimes N}_{\mathrm{restr}}(\mathcal{F}_\Sigma) \to \mathrm{sym}^{\times N}_{\mathrm{restr}}(\Sigma)$$

be the projection onto the base points. An N-point function on a Riemann surface Σ takes values $\langle\varphi\rangle_G$, where $\varphi \in \mathrm{sym}^{\boxtimes N}_{\mathrm{restr}}(\mathcal{F}_\Sigma)$ and $P(\varphi) \in \mathrm{sym}^{\times N}_{\mathrm{restr}}(\Sigma)$.

4 Definition of a rational conformal field theory

Three axioms are required to define the notion of a rational conformal field theory.

4.1 Axiom 1: Invariance under diffeomorphisms that preserve the conformal structure close to the respective base points

Using the previous notations, suppose $\Sigma \in \mathcal{M}_{g,0}$ and S is the oriented surface underlying Σ. Let f be an infinitesimal automorphism on

$$\mathrm{Met}_g(S) \times \mathrm{sym}^{\boxtimes N}_{\mathrm{restr}}(\mathcal{F}_\Sigma)\,.$$

On the first factor, f defines a diffeomorphic automorphism on $\mathrm{Met}_g(S)$ given by $G \mapsto G + \delta G$. Call this automorphism χ. On the second factor, f acts by $\varphi \mapsto \varphi + \delta_{\hat{f}}\,\varphi$, for some map $\hat{f}$. Our approach to CFT is through N-point functions $\langle\varphi\rangle_G$ for $\varphi \in \mathrm{sym}^{\boxtimes N}_{\mathrm{restr}}(\mathcal{F}_\Sigma)$ which restricts $\mathrm{Met}_g(S)$ to metrics G on S with $(G,\Sigma) \in \mathrm{M}_{g,0}$. Moreover, as we want to understand the change of $\langle\varphi\rangle_G$ under smooth variations of G, we only admit a specific class of diffeomorphisms which depends on the tuple $P(\varphi) = (p_1,\dots,p_N) \in \mathrm{sym}^{\times N}_{\mathrm{restr}}(\Sigma)$: We require that for $i = 1,\dots,N$ there exists a neighbourhood U_i of p_i in S such that after restriction to U_i, $\chi(G)|_{U_i}$ defines a metric in the conformal class defined by G or Σ. This allows to define the derivative of $\langle\varphi\rangle_G$ w.r.t. the metric G:

$$\langle\varphi\rangle_{G+\delta G} =: \langle\varphi\rangle_G + \delta_G\,\langle\varphi\rangle_G + O((\delta G)^2)\,, \tag{6}$$

It is easy to check that the map on the N-point function induced by f is given by

$$\langle\varphi\rangle_G \mapsto \langle\varphi + \delta_{\hat{f}}\,\varphi\rangle_{G+\delta G} = \langle\varphi\rangle_{G+\delta G} + \langle\varphi + \delta_{\hat{f}}\,\varphi\rangle_G - \langle\varphi\rangle_G + O(\delta_{\hat{f}}\varphi\cdot\delta G)\,,$$

using the defining properties of the state. The additive change to $\langle\varphi\rangle_G$ induced by f is

$$\Delta_f\langle\varphi\rangle_G := \langle\varphi + \delta_{\hat{f}}\,\varphi\rangle_{G+\delta G} - \langle\varphi\rangle_G\,. \tag{7}$$

Given a diffeomorphic automorphism f of S, let $\chi_f : \mathrm{Met}_g(S) \to \mathrm{Met}_g(S)$ be the natural induced diffeomorphism. By assumption, for $i = 1,\dots,N$, χ_f preserves the conformal structure on U_i. Thus f gives rise to germs $\langle U_i, f_i\rangle$ of conformal maps close to p_i, and thus by Proposition 9, to an automorphism $\Phi_F(f_i)$ of $\mathcal{F}_{\Sigma,p_i}$. We postulate that we have in eq. (7),

$$\Delta_f\langle\varphi\rangle_G = 0\,.$$

This means that for $\varphi = \varphi_1(p_1)\otimes\ldots\otimes\varphi_N(p_N)$,

$$\langle\Phi_F(f_1)\varphi_1(p_1)\otimes\ldots\otimes\Phi_F(f_N)\varphi_N(p_N)\rangle_{\chi_f(G)} = \langle\varphi_1(p_1)\otimes\ldots\otimes\varphi_N(p_N)\rangle_G\,.$$

4.2 Axiom 2: Einstein derivative

Let $\Sigma \in \mathcal{M}_{g,0}$ and $\mathcal{F}_\Sigma = \Phi_F(\Sigma)$. Let S be the oriented surface underlying Σ, with tangent bundle TS. Denote by $\mathrm{sym}^{\otimes 2}(T_\mathbb{R}\Sigma)$ the symmetric 2-fold tensor product of TS. We postulate that to every metric $G \in \mathrm{Met}(S)$, there exists an element $T \in \Gamma(\Sigma, \mathcal{F}_\Sigma \otimes \mathrm{sym}^{\otimes 2}(TS))$ such that for $\varphi \in \mathrm{sym}^{\boxtimes N}_{\mathrm{restr}}(\mathcal{F}_U)$, the derivative δ_G defined by (6) is given by

$$\delta_G\langle\varphi\rangle_G = \iint \langle(T,\delta G)\,\varphi\rangle_G\, d\mathrm{vol}_G\,. \tag{8}$$

Here $(\ ,\ )$ is the dual pairing, and $d\mathrm{vol}_G = \sqrt{|\det G_{\mu\nu}|}\,dx^0dx^1$ is the coordinate independent volume form.

4.3 Axiom 3: Trace Anomaly

Let $G \in \mathrm{Met}(S)$, and let $T \in \Gamma(\Sigma, \mathcal{F}_\Sigma \otimes \mathrm{sym}^{\otimes 2}(TS))$ be the corresponding element from Axiom 4.2. Let $\mathcal{R}_G$ be the scalar curvature of the Levi-Cività connection on S,

$$\mathcal{R}_G = G^{\kappa\lambda} R_{\kappa\lambda} \,.$$

Let T be the field from Subsection 4.2, and let $(.,.)$ be the dual pairing. We postulate that

$$(T, G) = -\frac{c}{48\pi}\,\mathcal{R}_G \,,$$

where $c \in \mathbb{R}$ is the central charge.

4.4 Definition of rational Conformal Field Theories

Definition 15. *Let F be a quasi-finite vector space. A (rational)* ***conformal field theory (CFT)*** *is a pair $(\Phi_F, \langle\ \rangle)$ where Φ_F is an F-functor and $\langle\ \rangle$ is a state such that Axiom 4.1, Axiom 4.2 and Axiom 4.3 are valid.*

5 Immediate Consequences of the Axioms

5.1 Conservation Law

According to Noether's theory, every continuous symmetry in a field theory gives rise to a conserved quantity. In a CFT, N-point functions are invariant under certain under diffeomorphisms (Axiom 4.1), and the corresponding conserved quantity is the energy momentum tensor.

$$\partial_\mu T^{\mu\nu} = 0 \,.$$

We shall explain the relationship with the Virasoro field $T(z)$ on $\Sigma \in \mathcal{M}_{g,0}$ and the induced conservation law. Let S be the oriented surface underlying Σ. Let $G \in \mathrm{Met}(S)$ and let $T \in \Gamma(\Sigma, \mathcal{F}_\Sigma \otimes \mathrm{sym}^{\otimes 2}(TS))$ be the corresponding Virasoro field. On any coordinate neighbourhood $U \subset \Sigma$, it is given by the energy momentum tensor

$$T|_U = \sum_{\mu,\nu=0}^{1} T^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \,.$$

Changing to complex coordinates $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$, we have [4]

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \,.$$

Lemma 1. *$T_{\mu\nu}$ satisfies the conservation law*

$$\nabla_\mu T^\mu{}_z = 0 \,.$$

Here ∇ is the covariant derivative of the Levi-Cività connection on S w.r.t. $G_{\mu\nu}$.

Proof. We have

$$\nabla_\mu T^\mu{}_z = \nabla_z T^z{}_z + \nabla_{\bar{z}} T^{\bar{z}}{}_z \,.$$

$T^z{}_z$ transforms like a scalar [7], so $\nabla_z T^z{}_z = \partial_z T^z{}_z$. Moreover, $\nabla_\mu G^{\mu\nu} = 0$ so

$$\nabla_{\bar{z}} T^{\bar{z}}{}_z = G^{z\bar{z}} \partial_{\bar{z}} T_{zz} \,.$$

This vanishes, since T_{zz} takes values in a holomorphic line bundle [7]. We conclude that

$$\nabla_\mu T^\mu{}_z = \partial_z T^z{}_z + G^{z\bar{z}}\partial_{\bar{z}} T_{zz} = 0 \, .$$

□

The Virasoro field does not depend on the specific metric on Σ, but only on the conformal class. $\langle T(x)\rangle \, (dx)^2$ defines a 0-cochain in the sheaf cohomology group of sheaf of holomorphic sections in $(T^*\Sigma)^{\otimes 2}$ associated to a complex analytic coordinate covering, but fails to satisfy the cocycle condition (i.e. to define a quadratic differential) when the coordinate changes induce the addition of a Schwarzian derivative term. The Schwarzian derivative, however, satisfies the 1-cocycle condition, and $\langle T(x)\rangle(dx)^2$ is known as projective connection.

Lemma 2. *[6] Suppose Σ has scalar curvature $\mathcal{R}$ = const. Let*

$$\frac{1}{2\pi} T(z) := T_{zz} - \frac{c}{24\pi} t_{zz} \, , \tag{9}$$

(with the analogous equation for $\bar{T}(\bar{z})$), where

$$t_{zz} := \left(\partial_z \Gamma^z{}_{zz} - \frac{1}{2}(\Gamma^z{}_{zz})^2\right).1 \, .$$

Here $\Gamma^z{}_{zz} = \partial_z \log G_{z\bar{z}}$ is the Christoffel symbol. We have

$$\partial_{\bar{z}} T(z) = 0 \, .$$

Proof. Direct computation shows that

$$\partial_{\bar{z}} t_{zz} = -\frac{1}{2} G_{z\bar{z}} \, \partial_z(\mathcal{R}.1) \, .$$

From the conservation law Lemma 1 follows

$$\begin{aligned} \partial_{\bar{z}} T_{zz} &= - G_{z\bar{z}} \, \partial_z T^z{}_z \\ &= - \frac{c}{48\pi} G_{z\bar{z}} \, \partial_z(\sqrt{G}\, \mathcal{R}.1) = \frac{c}{24\pi} \, \partial_{\bar{z}} t_{zz} \, . \end{aligned}$$

□

Thus for constant sectional curvature, $T(z)$ is a holomorphic quadratic differential.

Remark 16. *t_{zz} defines a projective connection: Under a holomorphic coordinate change, $z \mapsto w$ such that $w \in \mathcal{D}(S)$,*

$$t_{ww} \, (dw)^2 = t_{zz} \, (dz)^2 - S(w)(z).1 \, (dz)^2 \, ,$$

where $S(w)$ is the Schwarzian derivative,

$$S(w) = \frac{w'''}{w'} - \frac{3}{2}\left[\frac{w''}{w'}\right]^2 \, .$$

t_{zz} is known as the Miura transform *of the affine connection given by the differentials $\Gamma^z{}_{zz} dz$.*

$T(z)$ is the holomorphic field introduced in [10],[11].[1] For later reference, we note that from the transformation formula of t_{zz} and invariance of $T_{zz}(dz)^2$, the following transformation rule follows for $T(z)$: For a coordinate change $z \mapsto w$ with $w \in \mathcal{D}(S)$, we have

$$\hat{T}(w(z))\left[\frac{dw}{dz}\right]^2 = T(z) - \frac{c}{12}S(w)(z).1\,. \tag{10}$$

For infinitesimal $\varepsilon > 0$, consider the map $F_n : \Sigma \to \Sigma$ given by

$$F_n : z \mapsto \left(1 + \varepsilon f_n(z)\frac{\partial}{\partial z}\right)z = z(1 + \varepsilon z^n) \quad \text{for} \quad f_n(z) := z^{n+1}\,.$$

In particular, $F_n(0) = 0$.

Definition 17. *Suppose Σ has scalar curvature $\mathcal{R} = 0$. For $n \geq 0$, we define the map*

$$\delta_{F_n} : F \to F$$

as follows: For $\varphi(0) \in F = \mathcal{F}_{\Sigma,0}$,

$$\delta_{F_n}\varphi(0) := -\oint_\gamma f_n(z)T(z)\varphi(0)\,dz - \oint_\gamma \overline{f}_n(z)\overline{T}(z)\varphi(0)\,dz\,.$$

Here γ is any closed path not containing (but possibly enclosing) the argument of φ.

Claim 1. ***If φ is a holomorphic field, $\varphi \in F_{hol}$, then only the integral involving T contributes.***

Proof. The OPE of $\overline{T}(z_1) \otimes \varphi(z_2)$ has no singular part. Indeed, Laurent expansion of $\overline{T}(z_1)$ yields

$$\overline{T}(z_1) \otimes \varphi(z_2) = \sum_{n \geq n_0} (\bar{z}_1 - \bar{z}_2)^n A_n(z_2)$$

for the fields

$$A_n \left(= \frac{1}{n!}\frac{\partial^n T}{\partial z_2^n}|_{z_2}\varphi(z_2)\right)$$

which depend only on z_2, and the dependence is holomorphic. On the other hand, Laurent expansion of $\varphi(z_2)$ yields

$$\overline{T}(z_1) \otimes \varphi(z_2) = \sum_{m \geq m_0} (z_1 - z_2)^m B_m(z_1)\,,$$

where B_m depend holomorphically on z_1. The two expansions are incompatible unless the powers are non-negative. □

Claim 2. ***$T_{\mu\nu}$ does not depend on the specific metric, but only on the conformal class.*** *Thus $T(z)$ and $\overline{T}(z)$ defined for fixed $z = x^1 + ix^2$ by*

$$T_{\mu\nu}dx^\mu dx^\nu = T(z)dz^2 + \overline{T}(z)d\bar{z}^2 + \frac{c\mathcal{R}}{24\pi}dzd\bar{z}$$

define elements of $\Gamma(\Sigma, \mathcal{F}_\Sigma \boxtimes (T\Sigma)^{\otimes 2})$.

Proof. □

[1] Our notations differ from those used in [6]. Thus the standard field $T(z)$ in [6] equals $-T_{zz}$ in our exposition, and the field $\tilde{T}(z)$ in [6] equals $-\frac{1}{2\pi}T(z)$ here.

5.2 OPE of the Virasoro field

Application to the particular field $\varphi = T$ yields:

Claim 3. *The operator product expansion (OPE) of the Virasoro field reads*

$$T(z_1) \otimes T(z_2) \mapsto \frac{c/2}{(z_1 - z_2)^4}.1 + \frac{2T(z_2)}{(z_1 - z_2)^2} + reg.\,.$$

Proof. δ_{F_n} acts as a diffeomorphism on F. Under the transformation $z \mapsto F_n(z)$, T transforms according to eq. (10) as

$$\begin{aligned}\hat{T}(F_n(z)) &= (1 + \varepsilon f_n'(z))^{-2} \left(T(z) - \frac{c}{12} S(F_n)(z).1 \right) \\ &\approx (1 - 2\varepsilon f_n'(z)) \left(T(z) - \frac{c}{12} \frac{f_n'''(z)}{f_n'(z)}.1 \right) \quad \text{for} \quad |z| < 1\,,\ n \geq 0\,.\end{aligned}$$

On the other hand, $\hat{T}(F_n(z)) = T(z) + \varepsilon \delta_{F_n} T(z) + O(\varepsilon^2)$ where for γ enclosing $z = 0$,

$$\delta_{F_n} T(0) = -\oint_{\gamma(z=0)} f_n(z) T(z) T(0)\, dz\,.$$

So

$$\begin{aligned}-\oint_{\gamma(z=0)} f_n(z) T(z) T(0)\, dz &= -2 f_n'(0) \left(T(0) - \frac{c}{12} \frac{f_n'''(0)}{f_n'(0)}.1 \right) \\ &= -2(n+1) z^n|_{z=0} T(0) - \frac{c}{12} n(n^2 - 1) z^{n-2}|_{z=0}.1 \\ &= -2\delta_{n,0} T(0) - \frac{c}{2} \delta_{n,2}.1\,.\end{aligned}$$

Now on the l.h.s., $f_n = z^{n+1}$ sorts out the pole in the OPE of $T(z) \otimes T(0)$ of order $n + 2$. We let n run through $n = 0, 1, 2, \ldots$.

n	order of pole	term in OPE
0	2	$2T(0)$
2	4	$\frac{c}{2}.1$

□

The set $\{L_n\}_{n \in \mathbb{Z}}$ defined by

$$L_n := \oint \frac{T(z)}{z^{n-1}} \frac{dz}{2\pi i}$$

satisfies the Virasoro algebra

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0}\,, \tag{11}$$

a central extension of the Witt algebra (2).

6 The variation formula

6.1 The variation formula in the literature

We cosndier a surface S with metric $G_{\mu\nu}$. The effect on $\langle \mathbf{1} \rangle$ of a change $dG_{\mu\nu}$ in the metric is given by

$$d\langle \mathbf{1} \rangle = -\frac{1}{2} \iint dG_{\mu\nu} \, \langle T^{\mu\nu} \rangle \, \sqrt{G} \, dx^0 \wedge dx^1 \, . \tag{12}$$

Here $G := |\det G_{\mu\nu}|$, and $dvol_2 = \sqrt{G} \, dx^0 \wedge dx^1$ is the volume form which is invariant under base change. Eq. (12) generalises to the variation of N-point functions $\langle \varphi_1(x_1) \ldots \varphi_N(x_N) \rangle$ as follows: Suppose the metric is changed on an open subset $R \subseteq S$ of the surface S. Then

$$d\langle \varphi_1(x_1) \ldots \varphi_N(x_N) \rangle = -\frac{1}{2} \iint_S (dG_{\mu\nu}) \, \langle T^{\mu\nu} \varphi_1(x_1) \ldots \varphi_N(x_N) \rangle \, dvol_2 \, , \tag{13}$$

[18, eq. (12.2.2) on p. 360], see also [6, eq. (11)][2], provided that

$$x_i \notin R \, , \qquad \text{for } i = 1, \ldots, N \, . \tag{14}$$

Note that in order for the formula to be well-defined, $T_{\mu\nu} dx^\mu dx^\nu$ must be quadratic differential on S, i.e. one which transforms homogeneously under coordinate changes. The antiholomorphic contribution in eq. (13) is omitted. It is of course of the same form as the holomorphic one, up to complex conjugation.

Due to invariance of N-point functions under diffeomorphisms, $T_{\mu\nu}$ satisfies the conservation law Lemma 1.

A Weyl transformation $G_{\mu\nu} \mapsto \mathcal{W} G_{\mu\nu}$ changes the metric only within the respective conformal class. (In any chart (U, x) on S, such transformation is given by $G_{\mu\nu}(x) \mapsto h(x) G_{\mu\nu}(x)$ with $h(x) \neq 0$ on all of U.) The effect of a Weyl transformation on N-point functions is described by the trace of T (eq. (3) on p. 310 in [6]), which equals

$$T_\mu{}^\mu = T_z{}^z + T_{\bar{z}}{}^{\bar{z}} = 2 T_z{}^z = \frac{c}{24\pi} \mathcal{R}.1 \, , \tag{15}$$

([5], eq. (5.144) on page 140, which is actually true for the underlying fields). Here 1 is the identity field, and $\mathcal{R}$ is the scalar curvature of the Levi-Cività connection for ∇ on S. The non-vanishing of the trace (15) is referred to as the *trace* or *conformal anomaly*.

Since $T_\mu{}^\mu$ is a multiple of the unit field, the restriction (14) is unnecessary. Thus under a Weyl transformation $G_{\mu\nu} \mapsto \mathcal{W} G_{\mu\nu}$, all N-point functions change by the same factor Z (equal to $\langle \mathbf{1} \rangle$), given by

$$d \log Z = -\frac{c}{24\pi} \iint \mathcal{R} \, d\mathcal{W} \, dvol_2 \, .$$

While T_{zz} transforms as a two-form, it is not holomorphic. redefine the Virasoro field by Definition 9 to obtain a holomorphic field, but which as a result of the conformal anomaly, does not transform homogeneously in general.

[2]Note that both references introduce the Virasoro field with the opposite sign. Our sign convention follows e.g. [5], cf. eq. (5.148) on p. 140.

6.2 The concise statement and proof of the variation formula

Let S be a Riemann surface. We introduce

γ : one-dimensional smooth submanifold of S, topologically isomorphic to S^1,
R : a tubular neighbourhood of γ in S,
A : a vector field which conserves the metric on S and is holomorphic on R.

We think of $A \propto \frac{\partial}{\partial z} \in TR$ as an infinitesimal coordinate transformation

$$z \quad \mapsto \quad w(z) = \left(1 + \epsilon \frac{\partial}{\partial z}\right) z = z + \alpha(z)\,, \tag{16}$$

where $|\epsilon| \ll 1$.

Theorem 3. *Suppose S has scalar curvature $\mathcal{R} = 0$. Let φ be a holomorphic field on S. The effect of the transformation (16) on $\langle\varphi(w)\rangle$ is*

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \langle\varphi(w)\rangle = -i \oint_\gamma \langle T_{zz}\, \varphi(w)\rangle\, dz\,,$$

provided that

$$w \text{ does not lie on the curve } \gamma\,. \tag{17}$$

In particular, as w is not enclosed by γ, $\langle\varphi(w)\rangle$ doesn't change.

Proof. By property (17), the position of φ is not contained in a small tubular neighbourhood R of γ. Let

$$R \setminus \gamma = R_{\text{left}} \sqcup R_{\text{right}}$$

be the decomposition in connected parts left and right of γ (we assume γ has positive orientation). Let $W \subset S$ be an open set s.t.

$$\overline{W} \cap \gamma = \emptyset\,, \quad W \cup R = S\,.$$

We let $F : R \to [0, 1]$ be a smooth function s.t.

$$\begin{aligned} F &= 1 \quad \text{on} \quad R_{\text{left}} \cap W\,, \\ F &= 0 \quad \text{on} \quad R_{\text{right}} \cap W\,. \end{aligned}$$

Let ϵ be so small that $z \in W^c = S \setminus W$ implies $\exp(\epsilon F)(z) \in R$. Define a new metric manifold $(S^\epsilon, G^\epsilon_{z\bar{z}})$ by

$$\begin{aligned} S^\epsilon|_W &:= S|_W \\ G^\epsilon_{z\bar{z}}(z)\,|dz|^2 &:= G_{z\bar{z}}(\exp(\epsilon F)(z))\,|d\exp(\epsilon F)(z)|^2\,, \quad z \in W^c\,. \end{aligned}$$

We have

$$dG_{\mu\nu}T^{\mu\nu} = dG_{\bar{z}\bar{z}}T^{\bar{z}\bar{z}} + \text{antiholomorphic contributions} + \text{Weyl terms}\,,$$

where we disregard the antiholomorphic contributions $\sim T_{\bar{z}\bar{z}}$, and the Weyl terms are absent since by assumption $\mathcal{R} = 0$. Alternatively, we can describe the change in the metric by the map

$$|dz|^2 \mapsto |dz + \mu d\bar{z}|^2 = dz d\bar{z} + \mu d\bar{z} d\bar{z} + \dots\,,$$

where

$$\mu = \epsilon \partial_{\bar{z}} F + O(\epsilon^2)$$

is the Beltrami differential. Thus

$$dG_{\bar{z}\bar{z}} = 2G_{z\bar{z}}\, d\mu(z,\bar{z})\,.$$

Eq. (13) yields

$$\begin{aligned}
\frac{d\langle\varphi\rangle}{d\epsilon}|_{\epsilon=0} &= -\frac{1}{2}\iint_S \frac{\partial G_{\mu\nu}}{\partial\epsilon}|_{\epsilon=0}\,\langle T^{\mu\nu}\,\varphi\rangle\, dvol_2 \\
&= -\frac{i}{2}\iint_S 2G_{z\bar{z}}\,\frac{\partial\mu(z,\bar{z})}{\partial\epsilon}|_{\epsilon=0}\,(G^{z\bar{z}})^2\langle T_{zz}\,\varphi\rangle\, G_{z\bar{z}}\, dz\wedge d\bar{z} \\
&= i\iint_R (\partial_{\bar{z}}F)\,\langle T_{zz}\,\varphi\rangle\, d\bar{z}\wedge dz\,,
\end{aligned}$$

since $(G^{z\bar{z}})^k = (G_{z\bar{z}})^{-k}$ for $k\in\mathbb{Z}$. Here

$$\langle T_{zz}\,\varphi\rangle\, dz = \iota_A(\langle T_{zz}\,\varphi\rangle\,(dz)^2)$$

is the holomorphic 1-form given by the contraction of the holomorpic vector field $A = \frac{\partial}{\partial z}$ with the quadratic differential $\langle T_{zz}\,\varphi\rangle\,(dz)^2$, which is holomorphic on R. By Stokes' Theorem,

$$\begin{aligned}
\frac{d\langle\varphi\rangle}{d\epsilon}|_{\epsilon=0} &= i\iint_R \partial_{\bar{z}}\,(F\,\langle T_{zz}\,\varphi\rangle)\; d\bar{z}\wedge dz \\
&= i\oint_{W_R} F\,\langle T_{zz}\,\varphi\rangle\, dz + i\oint_{W_L} F\,\langle T_{zz}\,\varphi\rangle\, dz \\
&= -\,i\oint_{W_L} F\,\langle T_{zz}\,\varphi\rangle\, dz\,.
\end{aligned}$$

Here $W_R = N_R\cap\partial W$ and $W_L = N_L\cap\partial W$ are the left and right boundary, respectively, of W in R. We conclude that

$$\frac{d\langle\varphi\rangle}{d\epsilon}|_{\epsilon=0} = -i\oint_{W_L}\langle T_{zz}\,\varphi\rangle\, dz = -i\oint_{\gamma}\langle T_{zz}\,\varphi\rangle\, dz\,,$$

by holomorphicity on $R_{\text{left}}\cup\gamma$. □

Remark 18. *The construction is independent of F. When F approaches the discontinuous function defined by*

$$\begin{cases} F = 1 & \textit{on}\quad R_{left}\,, \\ F = 0 & \textit{on}\quad R_{right}\,, \end{cases}$$

we obtain a description of $(S^\epsilon, G^\epsilon_{\bar{z}\bar{z}})$ by cutting along γ and pasting back after a transformation by $\exp(\epsilon)$ *on the left.*

Remark 19. *The integral formula is similar to the conformal Ward identity in the literature [5] (in particular the so-called conformal Ward identity (5.46)). The exposition is not very clear, however, and may refer to global transformations, while we consider local coordinate transformations. Also, the contour of the integral is required to strictly enclose the position of any field contained in the N-point function, while we just require them not to lie on the contour of integration.*

There is a way to check the result of Theorem 3: Let φ be a holomorphic field whose position lies in a sufficiently small open set $U \subset S$ with boundary $\partial U = \gamma$. We can use a translationally invariant metric in U and corresponding coordinates $z, \bar{z}$. Then

$$T_{zz} = \frac{1}{2\pi} T(z)$$

in eq. (9). For $A = \frac{d}{dw}$, we have

$$\langle A\varphi(w)\ldots\rangle = \frac{1}{2\pi i}\oint_{\gamma}\langle T(z)\varphi(w)\ldots\rangle\, dz\,, \tag{18}$$

This can be seen in two ways.

1. Eq. (18) follows from the residue theorem for the OPE of $T(z) \otimes \varphi(w)$. Indeed, the Laurent coefficient of the first order pole at $z = w$ is $N_{-1}(T, \varphi)(w) = \partial_w \varphi$, which is holomorphic.

2. Alternatively, by Theorem 3,

$$\frac{d}{d\epsilon}\bigg|_{\epsilon}\langle\varphi(w+\epsilon)\ldots\rangle = \frac{1}{2\pi i}\oint_{\gamma}\langle T(z)\varphi(w)\ldots\rangle\, dz\,.$$

The two approaches are compatible!

6.3 Discussion of the metric

Let Σ_g be the genus g hyperelliptic Riemann surface

$$\Sigma_g : \quad y^2 = p(x)\,, \quad \deg p = n = 2g + 1\,.$$

Recall that x which varies over the Riemann sphere, defines a complex coordinate on Σ_g, outside the ramification points where we must change to the y coordinate. $\mathbb{P}^1_{\mathbb{C}}$ does not allow for a constant curvature metric but we shall define a metric on $\mathbb{P}^1_{\mathbb{C}}$ which is flat almost everywhere.

Suppose we consider a genus one surface with $n = 3$. By means of the isomorphism $\mathbb{P}^1_{\mathbb{C}} \cong \mathbb{C} \cup \{\infty\}$, we may identify the branch points of Σ_1 with points $X_1, X_2, X_3 \in \mathbb{C}$ and $X_4 = \{\infty\}$, respectively.

Let $\theta \gg 1$, but finite, such that in the flat metric of $\mathbb{C}$,

$$|X_i| < \theta\,, \quad i = 1, 2, 3\,.$$

We define $|X_4| := \infty$. For $\epsilon > 0$, define a metric

$$(ds(\epsilon))^2 = 2G_{z\bar{z}}(\epsilon)\, dz \otimes d\bar{z} \tag{19}$$

on $\mathbb{P}^1_{\mathbb{C}}$ by

$$2G_{z\bar{z}}(\epsilon) := \begin{cases} (1 + \epsilon\theta^2)^{-2} & \text{for} \quad |z| \leq \theta\,, \\ (1 + \epsilon z\bar{z})^{-2} & \text{for} \quad |z| \geq \theta\,. \end{cases}$$

The metric on Σ_1 is obtained by lifting.

Lemma 4. *In the disc $|z| \leq \theta$, the metric is flat, while in the area $|z| \geq \theta$, it is of Fubini-Study type of Gauss curvature $\mathcal{K} = 4\epsilon$.*

Proof. For $\rho = 2G_{z'\bar{z}'}(\epsilon)$ with

$$G_{z'\bar{z}'}(\epsilon) := \frac{1}{2\epsilon}(1 + z'\bar{z}')^{-2} \quad \text{for} \quad |z'| \geq \sqrt{\epsilon}\theta\,,$$

we have [7]

$$\mathcal{R} = \rho^{-1}(-4\partial_z\partial_{\bar{z}} \log \rho) = \epsilon(1 + z'\bar{z}')^2(8\partial_{z'}\partial_{\bar{z}'} \log(1 + z'\bar{z}')^2) = 8\epsilon\,,$$

and $\mathcal{R} = 2\mathcal{K}$. □

Definition 20. *Let Σ be a genus $g = 1$ Riemann surface with conformal structure defined by the position of the ramification points $\{X_i\}_{i=1}^3$ with finite relative distance on $\mathbb{P}^1_{\mathbb{C}}$. Let $G_{z\bar{z}}(\epsilon)$ be the metric defined by eq. (19). We define $\langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3,\epsilon,\theta}$ to be the zero-point function on $(\Sigma, G_{z\bar{z}}(\epsilon))$.*

By eq. (15) and the fact that on any surface, $\mathcal{R} = 2\mathcal{K}$,

$$T_{z\bar{z}} = \frac{c}{24\pi} G_{z\bar{z}} \mathcal{K}.1\,,$$

where 1 is the identity field. So according to eq. (12) we have for the 2-sphere S^2_θ of radius θ,

$$d \log\langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3,\epsilon,\theta} = \frac{c}{48\pi} \iint_{S^2_\theta} (d \log G_{z\bar{z}}(\epsilon))\, \mathcal{K}\, dvol_2\,.$$

Since $G(\epsilon) = (G_{z\bar{z}}(\epsilon))^2$, for $|z| > \theta$, the two-dimensional volume form is

$$dvol_2 = G_{z\bar{z}}(\epsilon)\, dz \wedge d\bar{z} = \frac{1}{2} \frac{\pi d(r^2)}{(1 + \epsilon r^2)^2}\,.$$

Now

$$d \log\langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3,\epsilon,\theta} = dI_{|z|<\theta} + dI_{|z|>\theta}\,,$$

where for $\varrho_0^2 := \epsilon\theta^2$, the integrals yield

$$\begin{aligned} dI_{|z|<\theta} &= -\frac{c\theta^2}{12}\, d(\epsilon)\, \frac{\varrho_0^2}{(1 + \varrho_0^2)^3}\,, \\ dI_{|z|>\theta} &= -\frac{c}{12}\, (d \log \epsilon) \int_{|\varrho|^2>\varrho_0^2} \frac{\varrho^2\, d(\varrho^2)}{(1 + \varrho^2)^3} = -\frac{c}{24}\, (d \log \epsilon)\, (1 + O(\varrho_0^4))\,. \end{aligned}$$

So for $|\varrho_0| \ll 1$,

$$\langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3,\epsilon,\theta} = \epsilon^{-\frac{c}{24}(1+O(\varrho_0^4))}\, Z\, \exp\left(-\frac{c}{12}\, \frac{\varrho_0^4}{(1 + \varrho_0^2)^3}\right)\,, \tag{20}$$

where $Z \in \mathbb{C}$ is an integration constant.

Variation of ϵ rescales the metric within the conformal class defined by the branch points. In the limit as $\epsilon \searrow 0$,

$$G_{z\bar{z}} := \lim_{\epsilon \searrow 0} G_{z\bar{z}}(\epsilon) = \frac{1}{2} \quad \text{for} \quad |z| < \infty\,, \tag{21}$$

(and is undefined for $|z| = \infty$). Thus $\mathbb{P}^1_{\mathbb{C}}$ becomes an everywhere flat surface except for the point at infinity, which is a singularity for the metric.

Definition 21. *Let Σ_1 be a genus $g = 1$ Riemann surface with conformal structure defined by the position of the ramification points $\{X_i\}_{i=1}^3$ with finite relative distance on $\mathbb{P}^1_{\mathbb{C}}$. Let $G_{z\bar{z}}$ be the metric on Σ defined by eq. (21). We define the zero-point function on $(\Sigma_1, G_{z\bar{z}})$ by*

$$\langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3} := \lim_{\rho_0 \searrow 0} \epsilon^{\frac{c}{24}(1+O(\varrho_0^4))} \langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3, \epsilon, \theta} \,.$$

Thus $\langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3} = Z$. We shall also write $\langle \mathbf{1} \rangle_{\text{sing.}}$ to emphasise distinction from the 0-point function on the flat torus $(\Sigma_1, |dz|^2)$, which we denote by $\langle \mathbf{1} \rangle_{\text{flat}}$.

Remark 22. *The reason for introducing ϵ and performing $\lim_{\epsilon \searrow 0}$ is the fact that the logarithm of the Weyl factor $\mathcal{W}$ is not defined for surfaces with a singular metric and infinite volume. We have*

$$d \log \frac{\langle \mathbf{1} \rangle_{sing.}}{\langle \mathbf{1} \rangle_{flat}} = d \log \mathcal{W} \,,$$

so $\mathcal{W}$ is determined only up to a multiplicative constant, which is infinite for $\epsilon = 0$.

Our method is available for any surface $\Sigma_g : y^2 = p(x)$ with $\deg p = n \geq 3$. When n is odd, the point at infinity is a non-distinguished element in the set of ramification points on Σ_g. We shall distribute the curvature of Σ_g evenly over these. Using the Gauss-Bonnet theorem, the total curvature is recovered as

$$\int_{\Sigma_g} \mathcal{K} \, dvol_2 = 2\pi \chi(\Sigma_g) = 4\pi(1-g) = 8\pi - 2\pi(2g+2) \,.$$

We interpret 8π as the contribution to the curvature from the $g = 0$ double covering and -2π from any branch point.

The method is now available for arbitrary genus $g \geq 1$ hyperelliptic Riemann surfaces and will in the following be checked against the case $g = 1$.

6.4 The main theorem

We now get to an algebraic description of the effect on an N-point function as the position of the ramification points of the surface is changed.

Theorem 5. *Let Σ_g be the hyperelliptic Riemann surface*

$$\Sigma_g : \quad y^2 = p(x) \,, \quad n = \deg p = 2g+1 \,,$$

with roots X_j. We equip the $\mathbb{P}^1_{\mathbb{C}}$ underlying Σ_g with the singular metric which is equal to

$$|dz|^2 \quad on\ \mathbb{P}^1_{\mathbb{C}} \setminus \{X_1 \ldots, X_n\} \,.$$

Let $\langle \ \rangle_{sing}$ be a state on Σ_g with the singular metric. We define a deformation of the conformal structure by

$$\xi_j = dX_j \quad for \quad j = 1, \ldots, n \,.$$

Let (U_j, z) be a chart on Σ_g containing X_j but no field position. We have

$$d\langle \varphi \ldots \rangle_{sing} = \sum_{j=1}^{n} \left(\frac{1}{2\pi i} \oint_{\gamma_j} \langle T(z) \varphi \ldots \rangle_{sing} \, dz \right) \xi_j \,, \tag{22}$$

where γ_j is a closed path around X_j in U_j.

Proof. On the chart (U, z), we have $\frac{1}{2\pi} T(z) = T_{zz}$ in eq. (9), outside the points which project onto one of the X_j for $j = 1, \ldots, n$ on $\mathbb{P}^1_{\mathbb{C}}$. Moreover, γ does not pick up any curvature for whatever path γ we choose. Since

$$d\langle \mathbf{1} \rangle_{\text{sing.}} = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial X_i} \langle \mathbf{1} \rangle_{\text{sing.}} ,$$

formula (22) follows from Theorem 3. □

7 Differential equation for N-point functions of the Virasoro field, for arbitrary genus

7.1 Notations

In the remainder of the paper, we will deal with very specific fields which will be distinguishable by the letter - $1, T, \vartheta, \psi$ - rather than by a lower index.

- To enhance readibility of the formulae, we shall denote $p(x), \vartheta(x), \ldots$ and $f(x, X_s)$ by $p_x, \vartheta_x, \ldots$ and f_{xX_s}. Instead of $f(x_1, x_2)$ and $p_{x_i}, \vartheta_{x_j}, \ldots$ we shall write f_{12} and $p_i, \vartheta_j, \ldots$, respectively. Thus

  $$f_{12} = \left(\frac{y_1 + y_2}{x_1 - x_2} \right)^2 .$$

  We shall avoid notations like $f_{x,y}$ and write instead f_x since $y = p(x)$. Subscripts will never denote derivatives. We we also use lower indices for the coefficents of Laurent series expansions, however, like

  $$a_k, \ \Theta_k, \ \Psi_k .$$

  These coefficients will not depend on position other than the reference point of the expansion, so the notatiion should be unambiguous.

- For a function f of x, we denote $f' = \frac{\partial}{\partial x} f$, and for $k \geq 3$, $f^{(k)} = \frac{\partial^k}{\partial x^k} f$. (However, in the notation $f^{(k)}$ we may include $k = 0, 1$.) We also write

  $$f'_{X_s} := \frac{d}{dx}|_{x=X_s} f_x .$$

  and for $\xi_s = dX_s$,

  $$d_{X_s} = \xi_s \frac{\partial}{\partial X_s} .$$

- We let

  $$\boxed{\omega_s := \sum_{t \neq s} \frac{\xi_s}{X_s - X_t}}$$

  and

  $$\omega := \sum_{s=1}^{n} \omega_s = \sum_{s=1}^{n} \sum_{t > s} \frac{\xi_s - \xi_t}{X_s - X_t} = \frac{1}{2} \sum_{\substack{t,s=1 \\ t \neq s}}^{n} \frac{\xi_s - \xi_t}{X_s - X_t} .$$

- By a pole at $x = 0$ we mean a $\frac{1}{x^m}$ singularity with $-m \in \mathbb{N} \setminus \{0\}$.
- For any rational function R of x, y with $y^2 = p(x)$, let $[R(x,y)]_{\text{no pole}}$ denote the projection of $R(x,y)$ onto those terms of $R(x,y)$ that have no pole at $x = X_s$ (but may have a squarte root singularity), where X_s is the image of a ramification point $(X_s, 0)$,on $\mathbb{P}^1_{\mathbb{C}}$ (a simple zero of $p = y^2$) specified in the context. Thus

$$[\vartheta(x)\vartheta_{X_s}]_{\substack{\text{no pole}\\ x=X_s}} = \lim_{x \to X_s} [\vartheta_x \vartheta_{X_s}]_{\text{no pole at } x = X_s} \;.$$

- The Schwarzian derivative of f w.r.t. x at x_0 is (assuming it is defined)

$$S(f_x)(x_0) := \frac{f^{(3)}_{x_0}}{f'_{x_0}} - \frac{3}{2}\left[\frac{f''_{x_0}}{f'_{x_0}}\right]^2 ,$$

where $f' = \frac{d}{dx} f$, etc.
- When using contour integrals, when P is a point on a surface S, we shall denote by γ_P a closed path in S that encloses the point P but does not pass through it.

7.2 Introduction of the auxiliary fields ϑ and ψ

We recall resp. generalise, a few definitions from [10] and [11]. Let ϑ be the field defined by

$$T_x\, p_x = \vartheta_x + \frac{c}{32}\frac{[p'_x]^2}{p_x}.1 \,. \tag{23}$$

Lemma 6. *Let $g \geq 1$. In the $(2,5)$ minimal model, the OPE for the field ϑ reads*

$$\vartheta_1 \otimes \vartheta_2 \;\mapsto\; \frac{c}{32} f_{12}^2 + \frac{1}{4} f_{12}(\vartheta_1 + \vartheta_2) + \psi_x + O((x_1 - x_2))\,, \tag{24}$$

where

$$\psi_x := -\frac{c}{480}[p'_x]^2 S(p_x).1 + \frac{1}{5}(p''_x\vartheta_x - \frac{1}{2}p'_x\vartheta'_x - p_x\vartheta''_x).1 \,. \tag{25}$$

Proof. From eqs (23) and (43),

$$\vartheta_x = \frac{[p'_x]^2}{4}\hat{T}_y + \frac{c}{12} p_x S(p_x)\,. \tag{26}$$

For brevity, we introduce the notation $S = S(p_x)(x)$ and for $i = 1, 2$, $S_i = S(p_x)(x_i)$. From the OPE for $\hat{T}_y$, using that in the $(2,5)$ minimal model, $\Phi_y = -\frac{1}{5}\partial_y^2\hat{T}_y$, we have

$$\begin{aligned}\vartheta_1 \otimes \vartheta_2 \mapsto \frac{[p'_1 p'_2]^2}{16}&\left(\frac{c}{2}\frac{1}{(y_1 - y_2)^4}.1 + \frac{\hat{T}_1 + \hat{T}_2}{(y_1 - y_2)^2} - \frac{1}{5}\partial_y^2\hat{T}_y\right) \\ &+ \frac{c}{6} p_x S\,\vartheta_x - \left(\frac{c}{12}p_x S\right)^2 .1 + O(y_1 - y_2)\,,\end{aligned} \tag{27}$$

where the expression on the r.h.s. of the arrow in line (27) reads

$$
\begin{aligned}
&\frac{c}{32}\frac{[p_1'p_2']^2}{(p_1-p_2)^4}(y_1+y_2)^4.1\\
&+\frac{1}{4}\frac{p_1'p_2'}{(p_1-p_2)^2}(y_1+y_2)^2\left\{(\vartheta_1-\frac{c}{12}p_1S_1)+(\vartheta_2-\frac{c}{12}p_2S_2)\right\}\\
&-\frac{[p_x']^4}{10}\left[\frac{1}{p_x'}\partial_x+\frac{2p_x}{p_x'}\partial_x\frac{1}{p_x'}\partial_x\right]\left(\frac{\vartheta_x-\frac{c}{12}p_xS.1}{[p_x']^2}\right),
\end{aligned}
$$

We use

$$
\begin{aligned}
&\frac{(p_1-p_2)^2}{p_1'p_2'}\\
&\quad=(x_1-x_2)^2\left(1-\frac{(x_1-x_2)^2}{12}(S_1+S_2)+\frac{(x_1-x_2)^4}{30}\left(\frac{S''(p_1)+S''(p_2)}{4}+\frac{S_1S_2}{3}\right)+O((x_1-x_2)^6)\right)
\end{aligned}
$$

(indeed, the l.h.s. is invariant under linear fractional transformations), and

$$
(y_1+y_2)^4=2(y_1+y_2)^2(p_1+p_2)-(p_1-p_2)^2\,.
$$

The expression on the r.h.s. of the arrow in line (27) becomes

$$
\begin{aligned}
&\frac{c}{32}f_{12}^2.1+\frac{1}{4}f_{12}(\vartheta_1+\vartheta_2)-\frac{c}{32}\left(\frac{p_1-p_2}{x_1-x_2}\right)^2\frac{S}{3} &(28)\\
&+\frac{c}{96}\left(\frac{y_1+y_2}{x_1-x_2}\right)^2(p_1-p_2)(S_1-S_2).1\\
&+\frac{c}{96}(y_1+y_2)^4\left(\left(\frac{S}{3}\right)^2-\frac{1}{5}\left(\frac{S''}{2}+\frac{S^2}{3}\right)\right).1\\
&+\frac{1}{4}(y_1+y_2)^2\frac{S}{3}\left(\vartheta_x-\frac{c}{12}p_xS.1\right)\\
&+\frac{1}{4}\left(\frac{y_1+y_2}{x_1-x_2}\right)^2(p_2'-p_1')\left(\frac{\vartheta_1-\frac{c}{12}p_1S.1}{[p_1']^2}-\frac{\vartheta_2-\frac{c}{12}p_2S.1}{[p_2']^2}\right)\\
&-\frac{[p_x']^3}{10}\partial_x\frac{\vartheta_x-\frac{c}{12}p_xS.1}{[p_x']^2}-\frac{p_x[p_x']^3}{5}\partial_x\frac{1}{p_x'}\partial_x\frac{\vartheta_x-\frac{c}{12}p_xS.1}{[p_x']^2} &(29)\\
&+O((x_1-x_2)^2)\,.
\end{aligned}
$$

Any term in the linear span of

$$
\begin{aligned}
&p_x^2S'',\ p_xp_x'S',\ p_xp_x''S,\ \frac{p_x^2p_x''}{p_x'}S',\ \frac{p_x^2[p_x'']^2}{[p_x']^2}S',\\
&\frac{p_xp_x^{(3)}}{p_x'}\vartheta_x,\ \frac{p_x[p_x'']^2}{[p_x']^2}\vartheta_x,\ \frac{p_xp_x''}{p_x'}\vartheta_x'
\end{aligned}
$$

must drop out from the OPE for ϑ by eq. (23), as $T(x)$ is regular at $p_x'=0$. Note that $[p_x']^2S$ is allowed. A combinatorial argument dealing with the number of factors of p_x and its derivatives, and the overall number of derivatives, (counted with sign), shows that every term must have a factor of p_x. The only term excluded from the list is $p_x\vartheta_x''$, which is allowed. We find that only lines (28) and (29) contribute to the OPE, where

$$-\frac{[p_x']^3}{10}\left[\partial_x + 2p_x\partial_x\frac{1}{p_x'}\partial_x\right]\frac{\vartheta_x}{[p_x']^2} = -\frac{1}{10}p_x'\vartheta_x' + \frac{1}{5}p_x''\vartheta_x - \frac{1}{5}p_x\vartheta_x'' + O(1/p_x')\,,$$
$$\frac{c}{12}\frac{[p_x']^3}{10}\left[\partial_x + 2p_x\partial_x\frac{1}{p_x'}\partial_x\right]\frac{p_xS}{[p_x']^2} = \frac{c}{120}[p_x']^2S + \frac{c}{24}p_xp^{(4)} + O(1/p_x')\,.$$

Collecting terms yields the claimed OPE. □

Claim 4. *For $k \in \mathbb{Z}$ and $1 \leq s \leq n$ fixed, we define an operator Θ_k on holomorphic fields by*

$$\Theta_k = \oint_{\gamma_{X_s}} \frac{\vartheta_x}{(x-X_s)^{k+1}}\frac{dx}{2\pi i}\,, \tag{30}$$

The operators Θ_k generate a non-commutative algebra which is equivalent to the Virasoro algebra (i.e., their respective commutation relations can be deduced from one another).

Remark 23. *We have*

$$\Theta_k = 0 \quad \textit{for}\; k < 0\,, \tag{31}$$

i.e. all N-point functions of Θ_k and $N-1$ holomorphic fields vanishes. So in the following, we shall always assume $k \in \mathbb{N}_0$.

Proof. We define a local coordinate $\check{y}_x$ for x near some ramification point in close distance to X_s, by

$$\check{y}_x^2 := (x - X_s)\,.$$

By the transformation formula eq. (10),

$$\frac{1}{4\check{y}^2}\check{T}_{\check{y}} = T_x - \frac{c}{32}\frac{1}{\check{y}^4}.1\,.$$

It is convenient to introduce $p_x =: (x-X_s)\hat{p}_x$. Thus by eq. (23),

$$\vartheta_x = \frac{1}{4}\check{T}_{\check{y}}\hat{p}_x - \frac{c}{32}\left(2\hat{p}_x' + \check{y}^2\frac{[\hat{p}_x']^2}{\hat{p}_x}\right).1\,,$$

(note that $\langle\vartheta_x\rangle$ is regular $x = X_s$ since $\hat{p}_{X_s} \neq 0$). This shows that

$$\oint_{X_s}\frac{\vartheta_x}{(x-X_s)^{k+1}}\frac{dx}{2\pi i} = \frac{1}{4}\oint_{X_s}\frac{\check{T}_{\check{y}}\hat{p}_x}{(x-X_s)^{k+1}}\frac{dx}{2\pi i} + \{\text{terms} \propto .1\}\,.$$

Set

$$\hat{p}_x = \sum_{\ell=0}^{n-1}\hat{a}_\ell(x-X_s)^\ell = \sum_{\ell=0}^{n-1}\hat{a}_\ell\check{y}^{2\ell}\,,$$

where $\hat{a}_\ell$ are constant in $\check{y}$. Then

$$\frac{1}{4}\oint_{\substack{X_s\\\text{twice}}}\frac{\check{T}_{\check{y}}\hat{p}_x}{\check{y}^{2k+2}}\frac{dx}{2\pi i} = \sum_{\ell=0}^{n-1}\hat{a}_\ell\oint_0\frac{\check{T}_{\check{y}}}{\check{y}^{2(k-\ell)+1}}\frac{d\check{y}}{2\pi i} = \sum_{\ell=0}^{n-1}\hat{a}_\ell\check{L}_{2(k-\ell+1)}\,,$$

where the $\check{L}_m$ satisfy the Virasoro algebra (11) (with $\check{L}_m$ in place of L_m). So the Θ_k satify the commutation relation

$$[\Theta_{k_1}, \Theta_{k_2}] = \frac{1}{4} \sum_{\ell_1,\ell_2=0}^{n-1} \hat{a}_{\ell_1} \hat{a}_{\ell_2} \left[\check{L}_{2(k_1-\ell_1+1)}, \check{L}_{2(k_2-\ell_2+1)}\right] . \tag{32}$$

(Note the factor of $1/2$ which accounts for circling X_s twice.) Inversely, for $|\sum_{\ell=1}^{n-1} \frac{\hat{a}_\ell}{a_0} \check{y}^{2\ell}| < 1$, that is, $|x - X_s| \ll 1$,

$$\frac{1}{\hat{p}_x} = \frac{1}{\hat{a}_0} \sum_{m=0}^{\infty} (-1)^m \left(\sum_{\ell=1}^{n-1} \frac{\hat{a}_\ell}{\hat{a}_0} \check{y}^{2\ell}\right)^m = \frac{1}{\hat{a}_0} \sum_{m=0}^{\infty} (-1)^m \sum_{m_1+m_2+\cdots+m_{n-1}=m} \frac{m!}{m_1! \ldots m_{n-1}!} \prod_{\ell=1}^{n-1} \left(\frac{\hat{a}_\ell}{\hat{a}_0} \check{y}^{2\ell}\right)^{m_\ell}$$

so

$$\begin{aligned}
\check{L}_{2(k+1)} &= \oint_0 \frac{\check{T}_{\check{y}}}{\check{y}^{2k+1}} \frac{d\check{y}}{2\pi i} \\
&= 2 \oint_{X_s} \frac{\vartheta_x}{(x - X_s)^{k+1} \hat{p}_x} \frac{dx}{2\pi i} + \{\text{terms} \propto .1\} \\
&= \frac{2}{\hat{a}_0} \sum_{m=0}^{\infty} (-1)^m \sum_{m_1+m_2+\cdots+m_{n-1}=m} \frac{m!}{m_1! \ldots m_{n-1}!} \left(\frac{\hat{a}_\ell}{\hat{a}_0}\right)^{\sum_{\ell=1}^{n-1} m_\ell} \Theta_{k-(\sum_{\ell=1}^{n-1} \ell \cdot m_\ell)} + \{\text{terms} \propto .1\} ,
\end{aligned}$$

and the commutation relation for the $\check{L}_n$ follows from that of the Θ_k. The sum is finite in practice, by eq. (31). □

Claim 5. *For $\ell \in \mathbb{Z}$ and $1 \le s \le n$ fixed, we define an operator Ψ_ℓ on holomorphic fields by*

$$\Psi_\ell = \oint_{\rho X_s} \frac{\psi_x}{(x - X_s)^{\ell+1}} \frac{dx}{2\pi i}$$

where ψ is the field defined in eq. (25). We have

$$\Psi_k = \sum_{m=0}^{k} \Theta_{k-m} \Theta_m + \textit{known correction terms} , \tag{33}$$

where Θ_k is given by eq. (30).

Proof. We have

$$\begin{aligned}
\Psi_k = &\oint_{\rho_{1,X_s}} \frac{1}{(x_1 - X_s)^{k+1}} \oint_{\rho_{2,x_1}} \frac{\vartheta_1 \vartheta_2}{x_2 - x_1} \frac{dx_2 dx_1}{(2\pi i)^2} &(34)\\
&+ \frac{c}{32} \oint_{\rho_{1,X_s}} \frac{1}{(x_1 - X_s)^{k+1}} \oint_{\rho_{2,x_1}} \frac{f_{12}^2 .1}{x_1 - x_2} \frac{dx_2 dx_1}{(2\pi i)^2} \\
&+ \frac{1}{4} \oint_{\rho_{1,X_s}} \frac{1}{(x_1 - X_s)^{k+1}} \oint_{\rho_{2,x_1}} f_{12} \frac{\vartheta_1 + \vartheta_2}{x_1 - x_2} \frac{dx_2 dx_1}{(2\pi i)^2} . &(35)
\end{aligned}$$

We address line (34). For $|x_1 - X_s| < |x_2 - X_s|$,

$$\frac{1}{x_2 - x_1} = \sum_{m=0}^{\infty} \frac{(x_1 - X_s)^m}{(x_2 - X_s)^{m+1}} ,$$

so by choosing a contour enclosing both x_1 and X_s,

$$\oint_{\rho_{1,X_s}} \frac{1}{(x_1 - X_s)^{k+1}} \oint_{\rho_{2,x_1}} \frac{\vartheta_1\vartheta_2}{x_2 - x_1} \frac{dx_2 dx_1}{(2\pi i)^2} = \sum_{m=0}^{k} \Theta_{k-m}\Theta_m \,.$$

In line (35), we replace accordingly

$$\frac{f_{12}}{x_2 - x_1} = (p_1 + p_2 + 2y_1y_2) \sum_{m=0}^{\infty} \frac{m(m+1)^2}{2} \frac{(x_1 - X_s)^m}{(x_2 - X_s)^{m+3}} \,.$$

Here for $x = x_1, x_2$, $p_x = (x - X_s)\hat{p}_x$. Taylor expansion of $\hat{p}_x$ about $x = X_s$ involves finitely many terms only. All occuring terms in line (35) are either known by reference to the Laurent coefficients $\check{L}_k$ of $\check{T}_{\check{y}}$, or they involve a square root of one of $x_1 - X_s$ and $x_2 - X_s$ and do not contribute. Eq. (33) follows. □

ϑ admits a Galois splitting

$$\vartheta_x = \vartheta_x^{[1]} + y\vartheta_x^{[y]} \,. \tag{36}$$

Note that $\vartheta^{[1]}$ and $\vartheta^{[y]}$ do in general not themselves define fields (except when one of the two equals ϑ_x). We define

$$\langle \vartheta_x \ldots \rangle =: \langle \vartheta_x^{[1]} \ldots \rangle + y\langle \vartheta_x^{[y]} \ldots \rangle \,.$$

Theorem 7. *Let $S(x_1, \ldots, x_N)$, $N \in \mathbb{N}$, be the set of oriented graphs with vertices $x_1, \ldots, x_N$, (not necessarily connected), subject to the following condition:*

> $\forall\ i = 1, \ldots, N$, *$x_i$ has at most one ingoing and at most one outgoing line,*
> *and if (x_i, x_j) is an oriented line connecting x_i and x_j then $i \neq j$.*

We have

$$\langle \vartheta_1 \ldots \vartheta_N \rangle = \sum_{\Gamma \in S(x_1, \ldots, x_N)} G(\Gamma) \,, \tag{37}$$

where for $\Gamma \in S(x_1, \ldots, x_N)$,

$$G(\Gamma) := \left(\frac{c}{2}\right)^{\sharp loops} \prod_{(x_i, x_j) \in \Gamma} \left(\frac{1}{4} f_{ij}\right) \left\langle \bigotimes_{k \in E_N{}^c} \vartheta_k \right\rangle_r \,,$$

where E_N are the endpoints.

Proof. Cf. Appendix, Section A. □

According to the graphical representation theorem, for x_1 close to x_2,

$$\boxed{\langle \vartheta_1\vartheta_2 \rangle = \frac{c}{32} f_{12}^2 \langle \mathbf{1} \rangle + \frac{1}{4} f_{12} \left(\langle \vartheta_1 \rangle + \langle \vartheta_2 \rangle \right) + \langle \vartheta_1\vartheta_2 \rangle_r} \,. \tag{38}$$

We will use the splitting

$$\langle \vartheta_1\vartheta_2 \rangle = \langle \vartheta_1\vartheta_2 \rangle^{[1]} + y_1 \langle \vartheta_1\vartheta_2 \rangle^{[y_1]} + y_2 \langle \vartheta_1\vartheta_2 \rangle^{[y_2]} + y_1y_2 \langle \vartheta_1\vartheta_2 \rangle^{[y_1y_2]} \,, \tag{39}$$

where e.g.

$$\left[\langle \vartheta_{X_s}\vartheta_x^{[1]} \rangle \right]_{\text{reg.}} = \left[\frac{c}{32} f_{X_s x}^2 \langle \mathbf{1} \rangle + \frac{1}{4} f_{X_s x} \left\{ \langle \vartheta_{X_s} \rangle + \langle \vartheta_x^{[1]} \rangle \right\} \right]_{\text{reg.}} + \langle \vartheta_{X_s}\vartheta_x^{[1]} \rangle_r \,, \tag{40}$$

$$\left[\langle \vartheta_{X_s}\vartheta_x^{[y]} \rangle \right]_{\text{reg.}} = \left[\frac{1}{4} f_{X_s x} \langle \vartheta_x^{[y]} \rangle \right]_{\text{reg.}} + \langle \vartheta_{X_s}\vartheta_x^{[y]} \rangle_r \,. \tag{41}$$

7.3 The differential equation for N-point functions of T

Lemma 8.

$$\left(d - \frac{c}{8}\,\omega\right)\langle T(x_1)\ldots T(x_N)\rangle = 2\sum_{s=1}^{n} \frac{\xi_s}{p'_{X_s}}\,\langle \vartheta_{X_s} T(x_1)\ldots T(x_N)\rangle\,. \tag{42}$$

Proof. We change to the y coordinate at $x = X_i$: We have $\frac{dy}{dx} = \frac{1}{2}y\frac{p'}{p}$, so

$$\hat{T}(y)\frac{[p']^2}{4p} = T(x) - \frac{c}{12}\left(S(p) + \frac{3}{8}\left(\frac{p'}{p}\right)^2\right).1\,. \tag{43}$$

Here $y(X_i) = 0$ and $S(p)$ is regular at $x = X_i$ (i.e. $p = 0$), so can be omitted from the contour integral.

$$\begin{aligned}
&d\langle T(x_1)\ldots T(x_N)\rangle \\
&= \frac{1}{2\pi i}\sum_{s=1}^{n}\left(\oint_{X_s}\langle T(x)T(x_1)\ldots T(x_N)\rangle\,dx\right)dX_s \\
&= \frac{1}{8\pi i}\sum_{s=1}^{n}\left(\oint_{\substack{X_s\\ \text{twice}}}\frac{[p'_x]^2}{p_x}\langle \hat{T}(y)T(x_1)\ldots T(x_N)\rangle\,dx\right)dX_s \\
&+ \frac{1}{2\pi i}\frac{c}{32}\langle T(x_1)\ldots T(x_N)\rangle\sum_{s=1}^{n}\left(\oint_{X_s}\left(\frac{p'}{p}\right)^2\,dx\right)dX_s\,,
\end{aligned}$$

In the first integral on the r.h.s. of the last identity, we wind around X_s twice.

Remark 24. *Note that the variation formula is compatible with the OPE, since d commutes with $\frac{c/2}{(x_1-x_2)^4}$ and $\frac{1}{(x_1-x_2)^2}$ in the (ordinary) Virasoro OPE. By induction, the singularities at $x_i = x_j$ for $1 \leq i < j \leq N$ are the same on both sides of the equation.*

We obtain, by eqs (43) and (23),

$$\begin{aligned}
\frac{1}{8\pi i}\oint_{\substack{X_s\\ \text{twice}}}\frac{[p']^2}{p}\langle \hat{T}(y)T(x_1)\ldots T(x_N)\rangle\,dx &= \frac{1}{2}p'_{X_s}\,\langle \hat{T}(0)T(x_1)\ldots T(x_N)\rangle \\
&= \frac{2}{p'_{X_s}}\,\langle \vartheta_{X_s}T(x_1)\ldots T(x_N)\rangle
\end{aligned}$$

Moreover,

$$\frac{1}{2\pi i}\oint_{X_s}\left(\frac{p'}{p}\right)^2\,dx = \frac{1}{2\pi i}\oint_{X_s}\left(\frac{1}{(x-X_s)^2} + \frac{2}{(x-X_s)}\sum_{j\neq s}\frac{1}{(x-X_j)}\right)dx = 4\sum_{j\neq s}\frac{1}{(X_s-X_j)}\,,$$

so

$$\frac{1}{2\pi i}\sum_{s=1}^{n}\xi_s\oint_{X_s}\left(\frac{p'}{p}\right)^2\,dx = 4\omega\,.$$

From this follows eq. (42). □

7.4 The differential equation for N-point functions of ϑ

Lemma 9.

$$\left(d - \frac{c}{8}\,\omega\right)\langle\vartheta_1\ldots\vartheta_N\rangle = 2\sum_{s=1}^{n}\frac{\xi_s}{p'_{X_s}}\langle\vartheta_{X_s}\vartheta_1\ldots\vartheta_N\rangle \tag{44}$$

$$+\langle\vartheta_1\ldots\vartheta_N\rangle\sum_{i=1}^{N}\frac{dp_i}{p_i} \tag{45}$$

$$-\frac{c}{16}\sum_{i=1}^{N}p'_i\,d\left(\frac{p'_i}{p_i}\right)\langle\vartheta_1\ldots\widehat{\vartheta}_i\ldots\vartheta_N\rangle\,, \tag{46}$$

Here

$$\begin{aligned}\frac{dp_x}{p_x} &= -\sum_{s=1}^{n}\frac{\xi_s}{x-X_s}\,,\\ d\left(\frac{p'}{p}\right) &= \sum_{s=1}^{n}\frac{\xi_s}{(x-X_s)^2}\,.\end{aligned} \tag{47}$$

Proof. By induction, cf. Appendix, Section B. □

Remark 25. *We show that the singularities on both sides of the differential equation in Lemma 9 are the same. By eq. (38), we have in line (44),*

$$\begin{aligned}f_{xX_s} &= \frac{p_x}{(x-X_s)^2}\\ &= \frac{p'_{X_s}}{x-X_s} + \frac{1}{2}p''_{X_s} + \frac{1}{6}p^{(3)}_{X_s}(x-X_s) + \frac{1}{24}p^{(4)}_{X_s}(x-X_s)^2 + O((x-X_s)^3)\,.\end{aligned}$$

So

$$\frac{1}{p'_{X_s}}f^2_{xX_s} = \frac{p'_{X_s}}{(x-X_s)^2} + \frac{p''_{X_s}}{x-X_s} + \frac{1}{4}\frac{[p''_{X_s}]^2}{p'_{X_s}} + \frac{1}{3}p^{(3)}_{X_s} + O(x-X_s)\,,$$

and the two singular terms cancel against corresponding terms of the sum in line (46), upon expansion of p'_x about $x_i = X_s$. Moreover, Taylor expansion in y about $x = X_s$ yields, in line (45),

$$-\frac{\vartheta_x}{x-X_s} = -\frac{\vartheta_{X_s}}{x-X_s} - \frac{y}{x-X_s}\vartheta^{[y]}_{X_s} - \frac{p_x}{x-X_s}\left(\frac{(\vartheta^{[1]})'_{X_s}}{p'_{X_s}} + y\frac{(\vartheta^{[y]})'_{X_s}}{p'_{X_s}}\right) + O(x-X_s)\,,$$

and in line (44),

$$\begin{aligned}\frac{1}{2}\frac{1}{p'_{X_s}}f_{xX_s}\{\vartheta_x+\vartheta_{X_s}\} &= \frac{\vartheta_{X_s}}{x-X_s} + \frac{1}{2}\frac{y}{x-X_s}\vartheta^{[y]}_{X_s}\\ &\quad+\frac{1}{2}\frac{p_x}{x-X_s}\frac{(\vartheta^{[1]})'_{X_s}}{p'_{X_s}} + \frac{1}{2}\frac{p''_{X_s}}{p'_{X_s}}\vartheta^{[1]}_{X_s}\\ &\quad+\frac{1}{2}y\frac{p_x}{x-X_s}\frac{(\vartheta^{[y]})'_{X_s}}{p'_{X_s}} + O(x-X_s)\,.\end{aligned} \tag{48}$$

So the first term on the r.h.s. of eq. (48) cancels against the corresponding summand in line (45). The second term on the r.h.s. of eq. (48) and in line (45), respectively, match the singularity on the l.h.s. of the differential equation, since

$$d_{X_s} y = \frac{\xi_s}{2} y \frac{\partial}{\partial X_s} \log p = -\frac{\xi_s}{2} \frac{y}{x - X_s} \,. \tag{49}$$

and

$$\begin{aligned} d_{X_s} \langle \vartheta_x \rangle = d_{X_s} \left(\langle \vartheta_x^{[1]} \rangle + y \langle \vartheta_x^{[y]} \rangle \right) &= d_{X_s} \langle \vartheta_x^{[1]} \rangle + (d_{X_s} y + y d_{X_s}) \langle \vartheta_x^{[y]} \rangle \\ &= d_{X_s} \langle \vartheta_x^{[1]} \rangle + y \left(d_{X_s} - \frac{\xi_s}{2} \frac{1}{x - X_s} \right) \langle \vartheta_x^{[y]} \rangle \,, \end{aligned} \tag{50}$$

upon expansion of $\langle \vartheta_x^{[y]} \rangle$ *about* $x = X_s$*. We conclude that the singularities on both sides of the differential equation are the same.*

Corollary 26. *For* $N = 1$, $1 \leq s \leq n$ *and* $\xi_i = \delta_{is}$,

$$\begin{aligned} \left(d_{X_s} - \frac{c}{8}\, \omega_s \right) \langle \vartheta_x \rangle = &- \xi_s \frac{c}{96} p'_{X_s} S(p_x)(X_s) \langle \mathbf{1} \rangle \\ &+ \frac{1}{2} \xi_s \frac{p''_{X_s}}{p'_{X_s}} \langle \vartheta_{X_s} \rangle \\ &- \frac{1}{2} \xi_s \frac{p_x}{x - X_s} \left(\frac{\langle (\vartheta^{[1]})'_{X_s} \rangle}{p'_{X_s}} + y \frac{\langle (\vartheta^{[y]})'_{X_s} \rangle}{p'_{X_s}} \right) \\ &+ \frac{2\xi_s}{p'_{X_s}} \langle \vartheta_{X_s} \vartheta_x \rangle_r + O(x - X_s) \,. \end{aligned} \tag{51}$$

Here $S(p_x)(X_s)$ *is the Schwarzian derivative of* p_x *w.r.t.* x *at* $x = X_s$*.*

Proof. The coefficient of $\langle \mathbf{1} \rangle$ in lines (44) and (46) for $N = 1$, to order $O(1)$ term at $x = X_s$, equals

$$\left[\frac{1}{p'_{X_s}} f^2_{x X_s} - p'_x \frac{\partial}{\partial X_s} \left(\frac{p'_x}{p_x} \right) \right]_{O(1)|_{x = X_s}} = -\frac{1}{6}\, p'_{X_s} S(p_x)(X_s) \,,$$

(cf. Remark 25). □

Higher genus requires more terms that are subsumed in $O(x - X_s)$.

8 Exact results for the $(2, 5)$ minimal model and arbitrary genus

8.1 Computation of ψ and $\langle \vartheta_{X_s} \vartheta_{X_s} \rangle_r$ for arbitrary genus

We consider the hyperelliptic genus $g \geq 1$ Riemann surface

$$\Sigma_g : \quad y^2 = p_x \,, \quad g \geq 1 \,,$$

$\deg p = n = 2g + 1$, with a distinguished ramification point $x = X_s$ which is a simple zero of p,

$$p_{X_s} = 0 \,, \quad p'_{X_s} \neq 0 \,.$$

Let

$$\mathcal{D} := -p\,\partial^2 - \frac{1}{2}p'\,\partial + p''\ ,$$

with $\partial = \frac{\partial}{\partial x}$. We have

$$\begin{aligned}\mathcal{D}\vartheta &= p''\vartheta^{[1]} - \frac{1}{2}p'\,(\vartheta^{[1]})' - p\,(\vartheta^{[1]})'' + y\left\{\frac{1}{2}p''\vartheta^{[y]} - \frac{3}{2}p'(\vartheta^{[y]})' - p(\vartheta^{[y]})''\right\}\\ &= p''\left(\vartheta^{[1]} + \frac{1}{2}y\vartheta^{[y]}\right) - \frac{1}{2}p'\left((\vartheta^{[1]})' + 3y(\vartheta^{[y]})'\right) - p\left((\vartheta^{[1]})'' + y(\vartheta^{[y]})''\right)\ .\end{aligned}$$

Thus in the $(2,5)$ minimal model, the Galois splitting of ϑ_x induces a Galois splitting of ψ_x by eq. (25). By means of the decomposition

$$\langle\vartheta_1\vartheta_2\rangle_r = \langle\vartheta_1^{[1]}\vartheta_2^{[1]}\rangle_r + y_1y_2\langle\vartheta_1^{[y]}\vartheta_2^{[y]}\rangle_r + y_1\langle\vartheta_1^{[y]}\vartheta_2^{[1]}\rangle_r + y_2\langle\vartheta_1^{[1]}\vartheta_2^{[y]}\rangle_r\ , \tag{52}$$

and by $\langle\vartheta_x\vartheta_x\rangle_r = \langle\psi_x\rangle$, the Galois splitting of ϑ induces a Galois splitting of Ψ,

$$\langle\psi_x\rangle = \langle\psi_x^{[1]}\rangle + y\langle\psi_x^{[y]}\rangle\ , \tag{53}$$

with

$$\begin{aligned}\langle\psi_x^{[1]}\rangle &= \langle\vartheta_x^{[1]}\vartheta_x^{[1]}\rangle_r + p_x\langle\vartheta_x^{[y]}\vartheta_x^{[y]}\rangle_r\\ \langle\psi_x^{[y]}\rangle &= 2\langle\vartheta_x^{[1]}\vartheta_x^{[y]}\rangle_r\ .\end{aligned}$$

Lemma 10. *For the Galois splitting eq. (53) of Ψ, we have*

$$\boxed{\langle(\psi^{[1]})'_{X_s}\rangle = \langle\vartheta_{X_s}(\vartheta^{[1]})'_{X_s}\rangle_r + \frac{1}{2}p'_{X_s}\langle\vartheta_{X_s}^{[y]}\vartheta_{X_s}^{[y]}\rangle_r}\ .$$

and

$$\boxed{\langle(\psi^{[y]})'_{X_s}\rangle = \langle\vartheta_{X_s}(\vartheta^{[y]})'_{X_s}\rangle_r}\ .$$

In the $(2,5)$ minimal model, these are known.

Proof. Cf. Appendix, Section C. □

Claim 6. *We assume the $(2,5)$ minimal model. We have*

$$\begin{aligned}\left[\langle\vartheta_{X_s}\vartheta_x^{[1]}\rangle\right]_{\substack{reg.\\ x=X_s}} &= \frac{c}{40}\left(\frac{1}{3}p'_{X_s}p^{(3)}_{X_s} + \frac{7}{16}[p''_{X_s}]^2\right)\langle\mathbf{1}\rangle + \frac{9}{20}p''_{X_s}\langle\vartheta_{X_s}\rangle + \frac{3}{20}p'_{X_s}\langle(\vartheta^{[1]})'_{X_s}\rangle\ ,\\ \left[\langle\vartheta_{X_s}\vartheta_x^{[y]}\rangle\right]_{\substack{reg.\\ x=X_s}} &= \frac{1}{4}\left(p''_{X_s}\langle\vartheta^{[y]}_{X_s}\rangle + p'_{X_s}\langle(\vartheta^{[y]})'_{X_s}\rangle\right) + \langle\vartheta_{X_s}\vartheta^{[y]}_{X_s}\rangle_r\ ,\end{aligned}$$

where

$$\langle\vartheta_{X_s}\vartheta^{[y]}_{X_s}\rangle_r = \frac{1}{2}\langle\psi^{[y]}_{X_s}\rangle$$

is known.

Proof. Cf. Appendix, Section D. □

8.2 The system of ODEs for $\langle \mathbf{1} \rangle$ and $\langle \vartheta_{X_s} \rangle$

Corollary 27. *Assume the* $(2,5)$ *minimal model. For* $g \geq 1$, *we have the system of ODEs*

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle \mathbf{1} \rangle = \frac{2\xi_s}{p'_{X_s}}\langle \vartheta_{X_s} \rangle\,, \tag{54}$$

$$\left(d_{X_s} - \frac{c}{8}\,\omega_s\right)\frac{\langle \vartheta_{X_s} \rangle}{p'_{X_s}} = \frac{77}{1200}\xi_s\,S(p_x)(X_s)\langle \mathbf{1} \rangle + \frac{2}{5}\xi_s\,\frac{p''_{X_s}}{p'_{X_s}}\frac{\langle \vartheta_{X_s} \rangle}{p'_{X_s}} + \frac{3}{10}\xi_s\,\frac{\langle (\vartheta^{[1]})'_{X_s} \rangle}{p'_{X_s}}\,, \tag{55}$$

where $S(p_x)(X_s)$ *is the Schwarzian derivative w.r.t.* x *evaluated at the position* $x = X_s$. *Moreover,*

$$\left(d_{X_s} - \frac{c}{8}\,\omega_s\right)\langle \vartheta^{[y]}_{X_s} \rangle = \frac{2\xi_s}{p'_{X_s}}\langle \vartheta_{X_s}\vartheta^{[y]}_{X_s} \rangle_r + \frac{1}{2}\xi_s\,\langle (\vartheta^{[y]})'_{X_s} \rangle\,.$$

where

$$\langle \vartheta_{X_s}\vartheta^{[y]}_{X_s} \rangle_r = \frac{1}{2}\langle \psi^{[y]}_{X_s} \rangle$$

is known.

Proof. The ODEs follow from Lemma 9 for $N = 0$ and $N = 1$, respectively, under the assumption $\xi_i = 0$ for $i \neq s$. For eq. (55), the ODE is given by Corollary 26. On the other hand, the l.h.s. is given by eq. (50). So the differential equations for the Galois even respectively the Galois odd part can be treated separately. For the two-point function in line (51), we use eq. (38) and the Galois splitting (52).

1. For the Galois-even part, we replace every copy of ϑ by $\vartheta^{[1]}$. We have seen that all singularities on the r.h.s. drop out in Remark 25. We argue that

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)|_{x=X_s}\langle \vartheta^{[1]}_x \rangle = \left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle \vartheta_{X_s} \rangle - \langle (\vartheta^{[1]})'_{X_s} \rangle\xi_s\,, \tag{56}$$

where $\langle \vartheta^{[1]}_{X_s} \rangle = \langle \vartheta_{X_s} \rangle$. Indeed, since both $\langle\quad\rangle$ and ϑ_{X_s} depend on X_s (and $\vartheta^{[1]}_x$ does not), set $\langle \vartheta_{X_s} \rangle = f(X_s, \vartheta_{X_s})$ for some function f. Then

$$\begin{aligned}
d_{X_s}\langle \vartheta_{X_s} \rangle &= \xi_s\frac{\partial}{\partial X_s}f(X_s, \vartheta_{X_s}) \\
&= \xi_s\frac{dx}{dX_s}\frac{\partial}{\partial x}|_{(x,y)=(X_s,\vartheta_{X_s})}f(x,y) + \xi_s\frac{dy}{dX_s}\frac{\partial}{\partial y}|_{(x,y)=(X_s,\vartheta_{X_s})}f(x,y) \\
&= d_{X_s}|_{x=X_s}\langle \vartheta^{[1]}_x \rangle + \xi_s\langle (\vartheta^{[1]})'_{X_s} \rangle\,.
\end{aligned}$$

From this and from eq. (25) follows

$$\begin{aligned}
\left(d_{X_s} - \frac{c}{8}\,\omega_s\right)\langle \vartheta_{X_s} \rangle = \xi_s\frac{c}{32}&\left(\frac{1}{2}\frac{[p''_{X_s}]^2}{p'_{X_s}} - \frac{1}{3}p^{(3)}_{X_s}\right)\langle \mathbf{1} \rangle \\
&+ \frac{1}{2}\xi_s\frac{p''_{X_s}}{p'_{X_s}}\langle \vartheta_{X_s} \rangle \\
&- \frac{1}{2}\xi_s\langle (\vartheta^{[1]})'_{X_s} \rangle + \xi_s\langle (\vartheta^{[1]})'_{X_s} \rangle \\
&- \frac{2\xi_s}{p'_{X_s}}\left(\frac{c}{480}\left(p'_{X_s}p^{(3)}_{X_s} - \frac{3}{2}[p''_{X_s}]^2\right)\langle \mathbf{1} \rangle + \frac{1}{10}\,p'_{X_s}\,\langle (\vartheta^{[1]})'_{X_s} \rangle - \frac{1}{5}\,p''_{X_s}\langle \vartheta_{X_s} \rangle\right)\,,
\end{aligned}$$

or

$$\left(d_{X_s} - \frac{c}{8}\,\omega_s\right)\langle\vartheta_{X_s}\rangle = -\xi_s \frac{7c}{480}\left(p^{(3)}_{X_s} - \frac{3}{2}\frac{[p''_{X_s}]^2}{p'_{X_s}}\right)\langle\mathbf{1}\rangle + \xi_s \frac{9}{10}\langle\vartheta_{X_s}\rangle + \xi_s \frac{3}{10}\langle(\vartheta^{[1]})'_{X_s}\rangle\,,$$

and thus eq. (55).

2. According to eqs (50) and (51), the differential equation for $\vartheta^{[y]}$ is given by

$$\left(d_{X_s} - \frac{c}{8}\,\omega_s\right)\langle\vartheta^{[y]}_x\rangle = \frac{2\xi_s}{p'_{X_s}}\langle\vartheta_{X_s}\vartheta^{[y]}_x\rangle_r - \frac{1}{2}\xi_s \frac{p_x}{x - X_s}\frac{\langle(\vartheta^{[y]})'\rangle}{p'_{X_s}} + O(x - X_s)\,.$$

Evaluating at $x = X_s$ and using the argument (56) yields the claimed formula.

This completes the proof. □

8.3 The LHS of the ODEs for $\langle\vartheta^{(k)}_{X_s}\ldots\rangle$, for arbitrary genus

Claim 7. *We consider the Galois-even part only. The l.h.s. of the differential equation for $N = 1$ reads*

$$\left(d_{X_s} - \frac{c}{8}\,\omega_s\right)\langle\vartheta_x\rangle = \sum_{k=0}^{n-2}\frac{1}{k!}(x - X_s)^k\,\left(d_{X_s}|_{x=X_s}\langle\vartheta^{(k)}_x\rangle - \frac{c}{8}\,\omega_s\langle\vartheta^{(k)}_{X_s}\rangle\right)\,,$$

For $N = 2$, $\langle\vartheta_1\vartheta_2\rangle$ is not differentiable at $x_2 = X_s$, but we have

$$\left(d_{X_s} - \frac{c}{8}\,\omega_s\right)\langle\vartheta_{x_1}\vartheta_{x_2}\rangle^{[1]} = \sum_{k=0}^{n-2}\frac{1}{k!\ell!}(x_1{-}X_s)^k(x_2{-}X_s)^\ell\left(\frac{\partial}{\partial X_s}|_{x_1,x_2=X_s}\langle\vartheta^{(k)}_{x_1}\vartheta^{(\ell)}_{x_2}\rangle^{[1]} - \frac{c}{8}\,\omega_s\langle\vartheta^{(k)}_{X_s}\vartheta^{(\ell)}_{X_s}\rangle^{[1]}\right).$$

We have

$$\frac{\partial}{\partial X_s}\langle\vartheta^{(k)}_{x_1}\vartheta^{(\ell)}_{X_s}\rangle^{[1]} = \frac{\partial}{\partial X_s}|_{x_2=X_s}\langle\vartheta^{(k)}_{x_1}\vartheta^{(\ell)}_{x_2}\rangle^{[1]} + \langle\vartheta^{(k)}_{x_1}\vartheta^{(\ell+1)}_{X_s}\rangle^{[1]}\,. \tag{57}$$

It is clear that this generalises to arbitrary finite N.

Proof. ($N = 1$) We consider the Galois-even part only. For $f(X_s, \vartheta^{(k)}_x) = \langle\vartheta^{(k)}_x\rangle_{\ldots,X_s,\ldots}$ and $k \geq 0$, we have

$$f(X_s, \vartheta^{(k)}_x) = f(X_s, \vartheta^{(k)}_{X_s} + \vartheta^{(k+1)}_{X_s}(x - X_s) + \frac{1}{2}\vartheta^{(k+2)}_{X_s}(x - X_s)^2 + \ldots)\,.$$

Since f is linear in its second argument, we have

$$f(X_s, \vartheta^{(k)}_x) = f(X_s, \vartheta^{(k)}_{X_s}) + f(X_s, \vartheta^{(k+1)}_{X_s})(x - X_s) + \frac{1}{2}f(X_s, \vartheta^{(k+2)}_{X_s})(x - X_s)^2 + \ldots\,.$$

Thus

$$\frac{\partial}{\partial X_s}|_{x=X_s} f(X_s, \vartheta^{(k)}_x) = \frac{\partial}{\partial X_s} f(X_s, \vartheta^{(k)}_{X_s}) - f(X_s, \vartheta^{(k+1)}_{X_s})\,.$$

i.e.

$$\frac{\partial}{\partial X_s}\langle\vartheta^{(k)}_{X_s}\rangle = \frac{\partial}{\partial X_s}|_{x=X_s}\langle\vartheta^{(k)}_x\rangle + \langle\vartheta^{(k+1)}_{X_s}\rangle$$

We apply this to the d_{X_s} derivative of

$$\langle\vartheta_x\rangle = \langle\vartheta_{X_s}\rangle + \langle(\vartheta')_{X_s}\rangle(x - X_s) + \frac{1}{2}\langle(\vartheta'')_{X_s}\rangle(x - X_s)^2 + \dots$$

We have

$$\begin{aligned}
\frac{\partial}{\partial X_s}\langle\vartheta_{X_s}\rangle &= \frac{\partial}{\partial X_s}|_{x=X_s}\langle\vartheta_x\rangle + \langle\vartheta'_{X_s}\rangle \\
\frac{\partial}{\partial X_s}\left\{\frac{1}{k!}\langle\vartheta^{(k)}_{X_s}\rangle(x - X_s)^k\right\} &= \frac{1}{k!}\left(\frac{\partial}{\partial X_s}|_{x=X_s}\langle\vartheta^{(k)}_x\rangle + \langle\vartheta^{(k+1)}_{X_s}\rangle\right)(x - X_s)^k \\
&\quad - \frac{1}{(k-1)!}\langle\vartheta^{(k)}_{X_s}\rangle(x - X_s)^{k-1}\,, \quad k \geq 1\,.
\end{aligned}$$

Thus in the expression for $d_{X_s}\langle\vartheta_x\rangle$, the terms $\xi_s\langle\vartheta'_{X_s}\rangle$ and $\frac{\xi_s}{k!}\langle\vartheta^{(k+1)}_{X_s}\rangle(x - X_s)^k$ drop out for $0 \leq k \leq \deg\langle\vartheta_x\rangle = n - 2$.

($N = 2$) For $f(X_s, \vartheta^{(k)}_x \dots) = \langle\vartheta^{(k)}_x \dots\rangle_{\dots,X_s,\dots}$ and $k \geq 0$, we have

$$f(X_s, \vartheta^{(k)}_{x_1}\vartheta^{(\ell)}_{x_2}) = \sum_{i,j}\frac{1}{j!}f(X_s, \vartheta^{(k)}_{x_1}\vartheta^{(\ell+j)}_{X_s})\,(x_2 - X_s)^j\,.$$

so

$$\frac{\partial}{\partial X_s}f(X_s, \vartheta^{(k)}_{x_1}\vartheta^{(\ell)}_{X_s}) = \frac{\partial}{\partial X_s}|_{x_2=X_s}f(X_s, \vartheta^{(k)}_{x_1}\vartheta^{(\ell)}_{x_2}) + f(X_s, \vartheta^{(k)}_{x_1}\vartheta^{(\ell+1)}_{X_s})\,.$$

or eq. (57). Also,

$$f(X_s, \vartheta^{(k)}_{x_1}\vartheta^{(\ell)}_{x_2}) = \sum_{i,j}\frac{1}{i!j!}f(X_s, \vartheta^{(k+i)}_{X_s}\vartheta^{(\ell+j)}_{X_s})\,(x_1 - X_s)^i(x_2 - X_s)^j\,.$$

Thus

$$\frac{\partial}{\partial X_s}f(X_s, \vartheta^{(k)}_{X_s}\vartheta^{(\ell)}_{X_s}) = \frac{\partial}{\partial X_s}|_{x_1=x_2=X_s}f(X_s, \vartheta^{(k)}_{x_1}\vartheta^{(\ell)}_{x_2}) + f(X_s, \vartheta^{(k+1)}_{X_s}\vartheta^{(\ell)}_{X_s} + \vartheta^{(k)}_{X_s}\vartheta^{(\ell+1)}_{X_s})$$

i.e.

$$\frac{\partial}{\partial X_s}\langle\vartheta^{(k)}_{X_s}\vartheta^{(\ell)}_{X_s}\rangle^{[1]} = \frac{\partial}{\partial X_s}|_{x_1=x_2=X_s}\langle\vartheta^{(k)}_{x_1}\vartheta^{(\ell)}_{x_2}\rangle^{[1]} + \langle\vartheta^{(k+1)}_{X_s}\vartheta^{(\ell)}_{X_s}\rangle^{[1]} + \langle\vartheta^{(k)}_{X_s}\vartheta^{(\ell+1)}_{X_s}\rangle^{[1]}\,.$$

We apply this to the d_{X_s} derivative of

$$\langle\vartheta^{(k)}_{x_1}\vartheta^{(\ell)}_{x_2}\rangle^{[1]} = \sum_{i,j}\frac{1}{i!j!}\langle\vartheta^{(k+i)}_{X_s}\vartheta^{(\ell+j)}_{X_s}\rangle^{[1]}\,(x_1 - X_s)^i(x_2 - X_s)^j\,.$$

We have for $k = 0$

$$\begin{aligned}
&\frac{\partial}{\partial X_s}\langle\vartheta_{X_s}\vartheta_{X_s}\rangle \\
&\quad = \frac{\partial}{\partial X_s}|_{x_1,x_2=X_s}\langle\vartheta_{x_1}\vartheta_{x_2}\rangle + \langle\vartheta'_{X_s}\vartheta_{X_s}\rangle + \langle\vartheta_{X_s}\vartheta'_{X_s}\rangle \\
&\frac{\partial}{\partial X_s}\left\{\frac{1}{\ell!}\langle\vartheta_{X_s}\vartheta^{(\ell)}_{X_s}\rangle\,(x_2 - X_s)^\ell\right\} \\
&\quad = \frac{1}{\ell!}\left(\frac{\partial}{\partial X_s}|_{x_1,x_2=X_s}\langle\vartheta_{x_1}\vartheta^{(\ell)}_{x_2}\rangle + \langle\vartheta'_{X_s}\vartheta^{(\ell)}_{X_s}\rangle + \langle\vartheta_{X_s}\vartheta^{(\ell+1)}_{X_s}\rangle\right)(x_2 - X_s)^\ell \\
&\qquad\qquad - \frac{1}{(\ell-1)!}\langle\vartheta_{X_s}\vartheta^{(\ell)}_{X_s}\rangle\,(x_2 - X_s)^{\ell-1}\,, \quad \ell \geq 1
\end{aligned}$$

and

$$\frac{\partial}{\partial X_s}\left\{\frac{1}{k!\ell!}\langle\vartheta^{(k)}_{X_s}\vartheta^{(\ell)}_{X_s}\rangle\,(x_1-X_s)^k(x_2-X_s)^\ell\right\}$$

$$\begin{aligned}
&= \frac{1}{k!\ell!}\left(\frac{\partial}{\partial X_s}|_{x_1,x_2=X_s}\langle\vartheta^{(k)}_{x_1}\vartheta^{(\ell)}_{x_2}\rangle + \langle\vartheta^{(k+1)}_{X_s}\vartheta^{(\ell)}_{X_s}\rangle + \langle\vartheta^{(k)}_{X_s}\vartheta^{(\ell+1)}_{X_s}\rangle\right)(x_1-X_s)^k(x_2-X_s)^\ell \\
&\qquad - \frac{1}{(k-1)!\ell!}\langle\vartheta^{(k)}_{X_s}\vartheta^{(\ell)}_{X_s}\rangle\,(x_1-X_s)^{k-1}(x_2-X_s)^\ell \\
&\qquad - \frac{1}{k!(\ell-1)!}\langle\vartheta^{(k)}_{X_s}\vartheta^{(\ell)}_{X_s}\rangle\,(x_1-X_s)^k(x_2-X_s)^{\ell-1}\,, \quad k,\ell\geq 1
\end{aligned}$$

Thus in the expression $d_{X_s}\langle\vartheta_{x_1}\vartheta_{x_2}\rangle$, the terms $\frac{\xi_s}{k!}\langle\vartheta^{(k+1)}_{X_s}\vartheta_{X_s}\rangle(x-X_s)^k$ and $\frac{1}{k)!\ell!}\langle\vartheta^{(k+1)}_{X_s}\vartheta^{(\ell)}_{X_s}\rangle(x_1-X_s)^k(x_2-X_s)^\ell$ drop out. □

8.4 The actual number of equations

Lemma 11. *We assume the* $(2,5)$ *minimal model. Let* Σ_g *have genus* $g\geq 1$ *and be defined by* $y^2 = p_x$ *where* $\deg p = n$*. Suppose* $\vartheta^{[y]}_x = 0$*. The number of differential equations required to specify* $\langle\mathbf{1}\rangle$ *equals a Fibonacci number.*

Proof. 1. Let P_n be the set of ascending chains, including the empty chain, of non-negative integer numbers $\leq n-3$,

$$i_1 < \ldots < i_k\,,\quad |i_j - i_{j+1}|\geq 2\,,\quad 1\leq j\leq k-1\,. \tag{58}$$

Let $F_n = \sharp P_n$. By considering partitions that do resp. do not contain the number n itself, we find

$$F_n = F_{n-1} + F_{n-2}.$$

Moreover, $F_1 = F_2 = 1$ (corresponding to $P_1 = P_2 = \{\emptyset\}$). Thus the F_n are the Fibonacci numbers. It remains to show that for $n = 2g+1$, F_n is the number of ODEs required.

2. For $g\geq 1$, $\langle\mathbf{1}\rangle$ is obtained by integrating the ODE

$$\mathcal{D}_s\langle\mathbf{1}\rangle = \frac{2\xi_s}{p'_{X_s}}\langle\vartheta_{X_s}\rangle\,,$$

(Lemma 9 for $N = 0$). $\langle\vartheta_x\rangle$ is a polynomial of degree $n-2$ whose leading coefficient only is known as a function of $\langle\mathbf{1}\rangle$ [11]. Indeed, for large x_1,

$$\langle\vartheta_x\ldots\rangle = -\frac{c}{32}(n^2-1)a_0x^{n-2}\langle\ldots\rangle + O(x^{n-3})\,, \tag{59}$$

where the dots stand for holomorphic fields. Thus $\langle\vartheta_x\rangle$ for x close to X_s is determined by $\langle\mathbf{1}\rangle$ and $\langle\vartheta^{(k)}_{X_s}\rangle$ for $k = 0,\ldots,n-3$. Assume now $\langle\vartheta_x\rangle$ for x close to X_s is given. The differential equation

$$\begin{aligned}
\left(d_{X_s} - \frac{c}{8}\,\omega_s\right)\langle\vartheta_x\rangle &= 2\frac{\xi_s}{p'_{X_s}}\langle\vartheta_{X_s}\vartheta_x\rangle \\
&\quad + \langle\vartheta_x\rangle\frac{dp_x}{p_x} \\
&\quad - \frac{c}{16}p'_x\,d\left(\frac{p'_x}{p_x}\right)\langle\mathbf{1}\rangle\,,
\end{aligned}$$

involves the two-point function $\langle\vartheta_{X_s}\vartheta_x\rangle$. By eq. (59), $\langle\vartheta_1\vartheta_1\rangle$ for x_1, x_2 close to X_s is determined by $\langle\vartheta_x\rangle$ (and thus by $\langle\mathbf{1}\rangle$ and $\langle\vartheta_x^{(k)}\rangle$) and by the derivatives $\langle\vartheta_1^{(k_1)}\vartheta_2^{(k_2)}\rangle$ for $0 \leq k_1, k_2 \leq n-3$. In the $(2,5)$ minimal model, the singular terms of $\langle\vartheta_1\vartheta_2\rangle$ and their derivatives are given by the OPE (using our previous knowledge of $\langle\vartheta_x^{(k)}\rangle$). Moreover, ψ_{X_s} is given by eq. (25) but while all Laurent coefficients Ψ_k from eq. (33) are known, the individual summands $\Theta_{k-m}\Theta_k$ of Ψ_k are not. By the commutation relations (32) for the Θ_k, an exchange of the factors in $\Theta_{i_k}\Theta_{i_{k+1}}$ within an N-point function gives rise to additional M-point functions with $M < N$, which has been dealt with before. Thus it is sufficient to consider pairs $\Theta_{k_i}\Theta_{k_{i+1}}$ with

$$k_{i+1} \geq k_i + 1\,.$$

which by knowledge of Ψ_k can be further restricted to

$$k_{i+1} \geq k_i + 2\,. \tag{60}$$

Proceeding inductively, the differential equation for the N-point function of the field ϑ_x involves an $(N+1)$-point function, and only the nonsingular terms of $\langle\vartheta_1^{(i_1)}\vartheta_2^{(i_2)}\ldots\vartheta_k^{(i_k)}\rangle$ for $1 \leq k \leq N+1$ are required at X_s. We can write them as $\langle\Theta_{i_1}\Theta_{i_2}\ldots\Theta_{i_k}\rangle$. By the commutation relations (32), we may assume condition (58) to hold.

The strictly monotonously increasing sequence (i_j) is bounded from above by $n-3$, which is the highest required order of derivative of ϑ_x. The procedure using the differential equation from Lemma 9 terminates and the number of N-point functions for which an equation is required is F_n.

□

For $k \geq 0$ and $N \geq 1$, we have

$$\begin{aligned}\frac{1}{k!}\Big(d_{X_s}\langle\vartheta_x^{(k)}\ldots\rangle|_{x=X_s} - \frac{c}{8}\,\omega_s\langle\vartheta_{X_s}^{(k)}\ldots\rangle\Big) &= 2\frac{\xi_s}{p'_{X_s}}\oint_{\gamma_{X_s}}\frac{\langle\vartheta_{X_s}\vartheta_x\ldots\rangle}{(x-X_s)^{k+1}}\,\frac{dx}{2\pi i}\\ &\quad - \xi_s\oint_{\gamma_{X_s}}\frac{\langle\vartheta_x\ldots\rangle}{(x-X_s)^{k+2}}\,\frac{dx}{2\pi i}\\ &\quad - \frac{c}{16}\,\xi_s\langle\ldots 1\rangle\oint_{\gamma_{X_s}}\frac{p'_x}{(x-X_s)^{k+3}}\,\frac{dx}{2\pi i}\,,\end{aligned}$$

where the dots stand for $N-1$ copies of ϑ_x. Using that

$$\oint_{\gamma_{X_s}}\frac{\langle\vartheta_{X_s}\vartheta_x\ldots\rangle}{(x-X_s)^{k+1}}\,\frac{dx}{2\pi i} = \oint_{\gamma_{X_s}}\frac{1}{(x_1-X_s)^{k+1}}\oint_{\gamma_{x_1}}\frac{\langle\vartheta_1\vartheta_2\ldots\rangle}{(x_2-x_1)}\,\frac{dx_2}{2\pi i}\frac{dx_1}{2\pi i}$$

and the OPE (24) for $\vartheta_1\otimes\vartheta_2$, the $(N+1)$-point function $\langle\vartheta_{X_s}\vartheta_x\ldots\rangle$ maps back to M-point functions with $M \leq N$. Thus

$$\begin{aligned}\oint_{\gamma_{X_s}}\frac{\langle\vartheta_{X_s}\vartheta_x\ldots\rangle}{(x-X_s)^{k+1}}\,\frac{dx}{2\pi i} &= \langle\Psi_k\ldots\rangle - \frac{c}{32}\langle\ldots\rangle\oint_{\rho_{X_s}}\frac{1}{(x_1-X_s)^{k+1}}\oint_{\rho_{x_1}}\frac{f_{12}^2}{x_1-x_2}\,\frac{dx_2dx_1}{(2\pi i)^2}\\ &\quad - \frac{1}{4}\oint_{\rho_{X_s}}\frac{1}{(x_1-X_s)^{k+1}}\oint_{\rho_{x_1}}f_{12}\frac{\langle\vartheta_1\ldots\rangle+\langle\vartheta_2\ldots\rangle}{x_1-x_2}\,\frac{dx_2dx_1}{(2\pi i)^2}\,.\end{aligned}$$

Here the dots stand for $N-1$ copies of ϑ, or of their derivatives.

Using the OPE (24) for $\vartheta_1 \otimes \vartheta_2$, the $(N+1)$-point function $\langle \vartheta_{X_s} \vartheta_x \rangle$ maps back to M-point functions with $M \leq N$, and to $\langle \psi_{X_s} \rangle$. The singuar terms are known in terms of $\langle \mathbf{1} \rangle$ and $\langle \vartheta_x \rangle$, which by our counting argument are supposed to be known. $\langle \psi_x \rangle$ for x close to X_s is determined by its Laurent coefficients Ψ_k.

For $N \geq 1$ and for $k \geq 0$, we obtain from the differential equation in Lemma 9

$$\begin{aligned}
\frac{1}{k!}\Big(d_{X_s}\langle \vartheta_1^{(k)} \vartheta_2 \ldots \vartheta_N \rangle|_{x_1 = X_s} - \frac{c}{8}\,\omega_s \langle \vartheta_{X_s}^{(k)} \vartheta_2 \ldots \vartheta_N \rangle\Big)& \\
= 2\frac{\xi_s}{p'_{X_s}} \oint_{\gamma_A} \frac{\langle \vartheta_{X_s} \vartheta_1 \ldots \vartheta_N \rangle}{(x_1 - X_s)^{k+1}}\,\frac{dx_1}{2\pi i}& \\
- \xi_s \sum_{i=1}^{N} \oint_{\gamma} \frac{1}{(x_i - X_s)} \frac{\langle \vartheta_1 \ldots \vartheta_N \rangle}{(x_1 - X_s)^{k+1}}\,\frac{dx_1}{2\pi i}& \\
- \frac{c}{16}\,\xi_s \sum_{i=1}^{N} \oint_{\gamma} \frac{p'_i}{(x_i - X_s)^2} \frac{\langle \vartheta_1 \ldots \widehat{\vartheta_i} \ldots \vartheta_N \rangle}{(x_1 - X_s)^{k+1}}\,\frac{dx_1}{2\pi i}&\,.
\end{aligned}$$

For $N \geq 1$ and for $k \geq 0$, we obtain from the differential equation in Lemma 9

$$\begin{aligned}
\frac{1}{k!}\Big(d_{X_s}\langle \vartheta_1^{(k_1)} \ldots \vartheta_N^{(k_N)} \rangle|_{x_i = X_s} - \frac{c}{8}\,\omega_s \langle \vartheta_{X_s}^{(k)} \vartheta_2 \ldots \vartheta_N \rangle\Big)& \\
= 2\frac{\xi_s}{p'_{X_s}} \oint_{\gamma_A} \frac{\langle \vartheta_{X_s} \vartheta_1 \ldots \vartheta_N \rangle}{(x_1 - X_s)^{k+1}}\,\frac{dx_1}{2\pi i}& \\
- \xi_s \sum_{i=1}^{N} \oint_{\gamma} \frac{1}{(x_i - X_s)} \frac{\langle \vartheta_1 \ldots \vartheta_N \rangle}{(x_1 - X_s)^{k+1}}\,\frac{dx_1}{2\pi i}& \\
- \frac{c}{16}\,\xi_s \sum_{i=1}^{N} \oint_{\gamma} \frac{p'_i}{(x_i - X_s)^2} \frac{\langle \vartheta_1 \ldots \widehat{\vartheta_i} \ldots \vartheta_N \rangle}{(x_1 - X_s)^{k+1}}\,\frac{dx_1}{2\pi i}&\,.
\end{aligned}$$

n	g	$\deg\langle\vartheta\rangle$ $= n-2$	$2g-2$	$0 \leq k_i \leq n-3$ $(k_I)_I$: $k_{i+1} \geq k_i + 2$	Diff. eq. $(d_s\|_{x=X_s})$ required for	$\sharp$ diff. eqs
3	1	1	0	$\emptyset, 0$	$\langle \mathbf{1} \rangle, \langle \vartheta \rangle$	2
4	1	2	0	$\emptyset, 0, \underline{1}$	$\langle \mathbf{1} \rangle, \langle \vartheta \rangle, \underline{\langle \vartheta' \rangle}$	3 ($\underline{2}$)
5	2	3	2	$\emptyset, 0, 1, 2$ 02	$\langle \mathbf{1} \rangle, \langle \vartheta \rangle, \langle \vartheta' \rangle, \langle \vartheta'' \rangle$ $\langle \vartheta\vartheta'' \rangle$	5
6	2	4	2	$\emptyset, 0, 1, 2, \underline{3}$ $02, 03, 13$	$\langle \mathbf{1} \rangle, \langle \vartheta \rangle, \langle \vartheta' \rangle, \langle \vartheta'' \rangle, \underline{\langle \vartheta^{(3)} \rangle}$ $\langle \vartheta\vartheta'' \rangle, \langle \vartheta\vartheta^{(3)} \rangle, \langle \vartheta'\vartheta^{(3)} \rangle$	8 ($\underline{7}$)
7	3	5	4	$\emptyset, 0, 1, 2, 3, 4$ $02, 03, 04, 13, 14, 24$ 024	$\langle \mathbf{1} \rangle, \langle \vartheta \rangle, \langle \vartheta' \rangle, \ldots, \langle \vartheta^{(4)} \rangle$ $\langle \vartheta\vartheta'' \rangle, \ldots, \langle \vartheta\vartheta^{(4)} \rangle, \langle \vartheta'\vartheta^{(3)} \rangle, \langle \vartheta'\vartheta^{(4)} \rangle, \langle \vartheta''\vartheta^{(4)} \rangle$ $\langle \vartheta\vartheta''\vartheta^{(4)} \rangle$	13

The counting of data required in the $(2,5)$ minimal model to establish the differential equation for $\langle \mathbf{1} \rangle$ in genus g. Here $\vartheta = \vartheta^{[1]}$. The underlined data are not actually required since for even n, the *two* first leading coefficients of $\langle \vartheta \rangle$ are known.

9 Explicit results for the $(2,5)$ minimal model and $g = 2$

9.1 The two-point function of ϑ for $g = 2$

Claim 8. *We assume $n = 5$ and the $(2,5)$ minimal model. We have the Galois splitting*

$$\langle\vartheta_1\vartheta_2\rangle = \langle\vartheta_1\vartheta_2\rangle^{[1]} + y_1y_2\langle\vartheta_1\vartheta_2\rangle^{[y_1y_2]}\,, \tag{61}$$

Here

$$\begin{aligned}\langle\vartheta_1\vartheta_2\rangle^{[1]} = {} & \frac{c}{4}\frac{p_1p_2}{(x_1-x_2)^4}\langle\mathbf{1}\rangle + \frac{c}{32}\frac{p_1'p_2'}{(x_1-x_2)^2}\langle\mathbf{1}\rangle + \frac{1}{2}\frac{p_1\langle\vartheta_2\rangle + p_2\langle\vartheta_1\rangle}{(x_1-x_2)^2}\\ & + \frac{7}{50}(p_1''\langle\vartheta_2\rangle + p_2''\langle\vartheta_1\rangle) + \frac{21c}{4000}p_1''p_2''\langle\mathbf{1}\rangle\\ & + B(x_1,x_2) + C\,(x_1-x_2)^2\,,\end{aligned} \tag{62}$$

where C is constant in position and

$$\begin{aligned}\beta_x := B(x,x) = {} & \left\{-\frac{7c}{960}p_x'p_x^{(3)} + \frac{91c}{16000}[p_x'']^2 + \frac{c}{8}\frac{p_xp_x^{(4)}}{24}\right\}\langle\mathbf{1}\rangle\\ & + \frac{1}{20}p_x\langle\vartheta_x''\rangle + \frac{3}{20}p_x'\langle\vartheta_x'\rangle - \frac{2}{25}p_x''\langle\vartheta_x\rangle\end{aligned}$$

is a polynomial of order 4*. Moreover, when multiplied by $\frac{2}{p'_{X_s}}$,*

$$\frac{c}{32}\frac{p_1'p_{X_s}'}{(x_1-X_s)^2}\langle\mathbf{1}\rangle$$

cancels against line (46) in the ODE given by Lemma 9 for $\langle\vartheta_1\rangle$ ($N = 1$) when $\xi_t = 0$ for $t \neq s$. In eq. (61),

$$\langle\vartheta_1\vartheta_2\rangle^{[y_1y_2]} = \frac{c}{8}\frac{p_1+p_2}{(x_1-x_2)^4}\langle\mathbf{1}\rangle + \frac{1}{2}\frac{\langle\vartheta_1\rangle + \langle\vartheta_2\rangle}{(x_1-x_2)^2} - \left(\frac{c}{16}\frac{p_1^{(4)}+p_2^{(4)}}{24} + \frac{1}{8}\langle\vartheta_1'' + \vartheta_2''\rangle\right).$$

We have

$$\begin{aligned}\langle\vartheta_1\vartheta_2\rangle = {} & \frac{c}{32}f_{12}^2\langle\mathbf{1}\rangle + \frac{1}{4}f_{12}\langle\vartheta_1 + \vartheta_2\rangle + \frac{1}{2}\langle\psi_1 + \psi_2\rangle\\ & + \left(\frac{c}{16}\frac{p_1^{(4)}+p_2^{(4)}}{24} + \frac{1}{8}\langle\vartheta_1'' + \vartheta_2''\rangle\right)\frac{(y_1-y_2)^2}{2} + O((x_1-x_2)^2)\end{aligned} \tag{63}$$

where terms up to $O((x_1-x_2)^2)$ are known.

Proof. For $n = 5$, $\Theta_x^{[y]} = 0$ and we have [10]

$$\langle\vartheta_x\rangle = \frac{1}{4}\Theta_x^{[1]} = -\frac{3c}{4}a_0x^3\langle\mathbf{1}\rangle + \frac{1}{4}\mathbf{A}_1x^2 + O(x)\,,$$

so as $x_1 \to \infty$,

$$\vartheta_1 = -\frac{3c}{4}a_0x_1^3.1 + O(x_1^2)\,,$$

and

$$\langle\vartheta_1\vartheta_2\rangle = -\frac{3c}{4}a_0x_1^3\langle\vartheta_2\rangle + O(x_1^2)\,. \tag{64}$$

On the other hand, in eq. (38),

$$f_{12} = \frac{p_1+p_2}{(x_1-x_2)^2} + 2y_1y_2\frac{1}{(x_1-x_2)^2}\,,$$
$$f_{12}^2 = \frac{p_1^2+6p_1p_2+p_2^2}{(x_1-x_2)^4} + 4y_1y_2\frac{p_1+p_2}{(x_1-x_2)^4}\,.$$

In eq. (39). the contribution

$$y_1\langle\vartheta_1\vartheta_2\rangle^{[y_1]} + y_2\langle\vartheta_1\vartheta_2\rangle^{[y_2]}$$

must be contained in $\langle\vartheta_1\vartheta_2\rangle_r$ and therefore equal zero for degree reasons. This yields eq. (61).

1. The terms $\propto y_1y_2$ in the singular part of eq. (38) are degree violating and must be compensated for by terms in $\langle\vartheta_1\vartheta_2\rangle_r$. $\langle\vartheta_1\vartheta_2\rangle^{[y_1y_2]}$ is a rational function in x_1 and x_2 which vanishes for $x_1 \to \infty$. Indeed, setting $\vartheta_x = \vartheta_x^{[1]} + y\,\vartheta_x^{[y]}$, we have

$$2y_2\langle\vartheta_1\vartheta_2^{[y]}\rangle = \langle\vartheta_1(\vartheta_2^{[1]}+y_2\vartheta_2^{[y]})\rangle - \langle\vartheta_1(\vartheta_2^{[1]}-y_2\vartheta_2^{[y]})\rangle = O(x_2^2)$$

in the large x_2 limit, by eq. (64). Thus

$$\langle\vartheta_1\vartheta_2\rangle^{[y_1y_2]} = \langle\vartheta_1^{[y]}\vartheta_2^{[y]}\rangle = O(x_2^{-0.5})\,. \tag{65}$$

As $x_1 \to \infty$,

$$[f_{12}^2]^{[y_1y_2]} = 4a_0(x_1+4x_2) + 4a_1 + O(x_1^{-1})\,,$$

and

$$[f_{12}\langle\vartheta_1\rangle]^{[y_1y_2]} = [f_{12}]^{[y_1y_2]}\langle\vartheta_1\rangle = -\frac{3c}{2}a_0(x_1+2x_2)\langle\mathbf{1}\rangle + \frac{1}{2}\mathbf{A}_1 + O(x_1^{-1})\,.$$

We conclude that for $x_1 \to \infty$,

$$\left[\frac{c}{32}f_{12}^2\langle\mathbf{1}\rangle + \frac{1}{4}f_{12}\langle\vartheta_1\rangle\right]^{[y_1y_2]} = -\frac{c}{4}a_0(x_1+x_2)\langle\mathbf{1}\rangle + \frac{c}{8}a_1\langle\mathbf{1}\rangle + \frac{1}{8}\mathbf{A}_1 + O(x_1^{-1})\,.$$

Thus we compensate by addition of y_1y_2C, where

$$C = -\left(\frac{c}{16}\frac{p_1^{(4)}+p_2^{(4)}}{24} + \frac{1}{8}\langle\vartheta_1''+\vartheta_2''\rangle\right) = \frac{c}{4}a_0(x_1+x_2)\langle\mathbf{1}\rangle - \frac{1}{8}\mathbf{A}_1\,. \tag{66}$$

2. The term $\frac{c}{32}f_{12}^2\langle\mathbf{1}\rangle$:

$$\begin{aligned}\frac{p_1^2+6p_1p_2+p_2^2}{(x_1-x_2)^4} &= \frac{8p_1p_2}{(x_1-x_2)^4} + \frac{1}{(x_1-x_2)^2}\left(\frac{p_1-p_2}{x_1-x_2}\right)^2\\ &= \frac{8p_1p_2}{(x_1-x_2)^4} + \frac{p_1'p_2'}{(x_1-x_2)^2} + \frac{1}{4}p_1''p_2'' - \frac{1}{12}(p_1'p_2^{(3)}+p_2'p_1^{(3)}) + O((x_1-x_2)^2)\\ &= \frac{8p_1p_2}{(x_1-x_2)^4} + \frac{p_1'p_2'}{(x_1-x_2)^2} - \frac{1}{12}\left\{\left(p_1'p_2^{(3)}+p_2'p_1^{(3)}\right) - \frac{3}{2}\left([p_1'']^2+[p_2'']^2\right)\right\} + O((x_1-x_2)^2)\end{aligned}$$

The term $\frac{1}{4}f_{12}\,(\vartheta_1+\vartheta_2)$:

$$\begin{aligned}\frac{(p_1+p_2)(\langle\vartheta_1\rangle+\langle\vartheta_2\rangle)}{(x_1-x_2)^2} &= 2\,\frac{p_1\langle\vartheta_2\rangle+p_2\langle\vartheta_1\rangle}{(x_1-x_2)^2}+\frac{p_1-p_2}{x_1-x_2}\frac{\langle\vartheta_2\rangle-\langle\vartheta_1\rangle}{x_2-x_1}\\ &= 2\,\frac{p_1\langle\vartheta_2\rangle+p_2\langle\vartheta_1\rangle}{(x_1-x_2)^2}+\frac{1}{2}(p_1'\langle\vartheta_2'\rangle+p_2'\langle\vartheta_1'\rangle)+O((x_1-x_2)^2)\,.\end{aligned}$$

Introduce

$$\tilde\vartheta_x := \vartheta_x+\frac{3c}{80}p_x''.1\,,\quad \deg\tilde\vartheta_x=2\,.$$

Correcting the order violating singular terms and omitting the order violating regular terms in the previous expansions yields

$$\begin{aligned}\langle\tilde\vartheta_1\tilde\vartheta_2\rangle &= [\langle\vartheta_1\vartheta_2\rangle]_{\text{order}\le 2}+B(x_1,x_2)+C(x_1-x_2)^2\\ &= \left[\frac{c}{32}f_{12}^2\langle\mathbf{1}\rangle+\frac{1}{4}f_{12}\,(\vartheta_1+\vartheta_2)+\langle\vartheta_1\vartheta_2\rangle_r\right]_{\text{order}\le 2}+B(x_1,x_2)+C(x_1-x_2)^2\\ &= \frac{c}{4}\frac{p_1p_2}{(x_1-x_2)^4}\langle\mathbf{1}\rangle+\frac{c}{32}\frac{p_1'p_2'}{(x_1-x_2)^2}\langle\mathbf{1}\rangle+\frac{1}{2}\,\frac{p_1\langle\vartheta_2\rangle+p_2\langle\vartheta_1\rangle}{(x_1-x_2)^2}+y_1y_2\left(\frac{c}{8}\frac{p_1+p_2}{(x_1-x_2)^4}+\frac{1}{2}\frac{\langle\vartheta_1+\vartheta_2\rangle}{(x_1-x_2)^2}\right)\\ &\quad -\frac{1}{40}(p_1''\langle\vartheta_2\rangle+p_2''\langle\vartheta_1\rangle)-\frac{3c}{3200}p_1''p_2''-y_1y_2\left(\frac{c}{16}\frac{p_1^{(4)}+p_2^{(4)}}{24}+\frac{1}{8}\langle\vartheta_1''+\vartheta_2''\rangle\right)\\ &\quad +B(x_1,x_2)+C(x_1-x_2)^2\,,\end{aligned}$$

where $B(x_1,x_2)$ is a symmetric polynomial in x_1 and x_2 of order $\mathrm{ord}_iB(x_1,x_2)=2$ for $i=1,2$. (The second line contains the order correcting terms of the singular terms.) On the other hand, by the OPE (24) and by eq. (25),

$$\begin{aligned}\langle\tilde\vartheta_1\tilde\vartheta_2\rangle &= \langle\vartheta_1\vartheta_2\rangle+\frac{3c}{80}(p_1''\langle\vartheta_2\rangle+p_2''\langle\vartheta_1\rangle)+\left(\frac{3c}{80}\right)^2p_1''p_2''\\ &= \frac{c}{32}f_{12}^2\langle\mathbf{1}\rangle+\frac{1}{4}f_{12}\,(\langle\vartheta_1\rangle+\langle\vartheta_2\rangle)+\langle\psi_1\rangle+O(x_1-x_2)+\left(\frac{3c}{80}\right)^2p_1''p_2''+\frac{3c}{80}(p_1''\langle\vartheta_2\rangle+p_2''\langle\vartheta_1\rangle)\\ &= \frac{c}{4}\frac{p_1p_2}{(x_1-x_2)^4}\langle\mathbf{1}\rangle+\frac{c}{32}\frac{p_1'p_2'}{(x_1-x_2)^2}\langle\mathbf{1}\rangle+\frac{1}{2}\,\frac{p_1\langle\vartheta_2\rangle+p_2\langle\vartheta_1\rangle}{(x_1-x_2)^2}+y_1y_2\left(\frac{c}{8}\frac{p_1+p_2}{(x_1-x_2)^4}+\frac{1}{2}\frac{\langle\vartheta_1+\vartheta_2\rangle}{(x_1-x_2)^2}\right)\\ &\quad +\frac{3c}{40}p_1''\langle\vartheta_1\rangle+\frac{9c^2}{6400}[p_1'']^2\\ &\quad -\frac{7c}{960}(p_1'p_1^{(3)}-\frac{3}{2}[p_1'']^2)+\frac{1}{5}p_1''\langle\vartheta_1\rangle+\frac{3}{20}p_1'\langle\vartheta_1'\rangle-\frac{1}{5}p_1\langle\vartheta_1''\rangle+O(x_1-x_2)\end{aligned}$$

By comparison, we obtain

$$\begin{aligned}\langle\vartheta_1\vartheta_2\rangle^{[1]} &= \frac{c}{4}\frac{p_1p_2}{(x_1-x_2)^4}\langle\mathbf{1}\rangle+\frac{c}{32}\frac{p_1'p_2'}{(x_1-x_2)^2}\langle\mathbf{1}\rangle+\frac{1}{2}\,\frac{p_1\langle\vartheta_2\rangle+p_2\langle\vartheta_1\rangle}{(x_1-x_2)^2}\\ &\quad -\left(\frac{3c}{80}+\frac{1}{40}\right)(p_1''\langle\vartheta_2\rangle+p_2''\langle\vartheta_1\rangle)-\left(\left(\frac{3c}{80}\right)^2+\frac{3c}{3200}\right)p_1''p_2''\langle\mathbf{1}\rangle\\ &\quad +B(x_1,x_2)+C\,(x_1-x_2)^2\,,\end{aligned}$$

where

$$B(x,x) = -\frac{7c}{960}p'_x p^{(3)}_x + \left(\frac{9c^2}{6400} + \frac{3c}{3200} + \frac{7c}{640}\right)[p''_x]^2 + \frac{c}{8}\frac{p_x p^{(4)}_x}{24}$$
$$+ \frac{1}{20}p_x\langle\vartheta''_x\rangle + \frac{3}{20}p'_x\langle\vartheta'_x\rangle + \left(\frac{1}{5} + \frac{3c}{40} + \frac{1}{20}\right)p''_x\langle\vartheta_x\rangle$$

and thus as required. $B(x,x)$ is a polynomial of order 4, though it is not manifestly so.

□

Remark 28. *We have*

$$\begin{array}{rcl} \beta_x & = & B(x,x) \\ \beta'_x & = & (\partial_1 B + \partial_2 B)|_{x_1=x_2=x} \\ \frac{1}{6}\beta''_x & = & \frac{1}{2}(\partial_1^2 B + \partial_2^2 B)|_{x_1=x_2=x} \\ & = & 2\partial_1\partial_2 B|_{x_1=x_2=x} \\ \frac{1}{6}\beta^{(3)}_x & = & \frac{1}{2}(\partial_2^2\partial_1 B + \partial_1^2\partial_2 B)|_{x_1=x_2=x} \end{array}$$

For evaluating the contour integral, the corresponding non-symmetric formulations are more suitable,

$$\begin{array}{rcl} \beta_x & = & B(x,x) \\ \frac{1}{2}\beta'_x & = & (\partial_2 B)(x,x) \\ \frac{1}{6}\beta''_x & = & \partial_2^2 B(x,x) \\ \frac{1}{6}\beta^{(3)}_x & = & (\partial_1^2\partial_2 B)(x,x) \end{array}$$

Claim 9. *We have*

$$\beta'_x = \left\{\frac{49c}{12000}p''_x p^{(3)}_x - \frac{c}{480}p'_x p^{(4)}_x + \frac{c}{300}p_x p^{(5)}\right\}\langle\mathbf{1}\rangle$$
$$+ \frac{1}{5}p'_x\langle\vartheta''_x\rangle + \frac{7}{100}p''_x\langle\vartheta'_x\rangle - \frac{2}{25}p^{(3)}_x\langle\vartheta_x\rangle$$

and

$$\beta^{(3)}_x = \left(\frac{61c}{6000}p^{(3)}_x p^{(4)}_x - \frac{23c}{1600}p''_x p^{(5)}\right)\langle\mathbf{1}\rangle$$
$$+ \frac{13}{50}p^{(3)}_x\langle\vartheta''_x\rangle - \frac{9}{100}p^{(4)}_x\langle\vartheta'_x\rangle - \frac{2}{25}p^{(5)}_x\langle\vartheta_x\rangle$$

Proof. Direct computation, using that

$$\langle\vartheta^{(3)}\rangle = -\frac{3c}{80}p^{(5)}\langle\mathbf{1}\rangle\,, \quad p^{(5)} = 120a_0\,. \tag{67}$$

□

9.2 The system of exact ODEs for $g = 2$ ($n = 5$)

Example 29. *Let $g = 2$ ($n = 5$) and $a_0 = \frac{1}{5!}p^{(5)}_{X_s}$.*

$$f_{xX_s} = \frac{p_x}{(x - X_s)^2}$$
$$= \frac{p'_{X_s}}{x - X_s} + \frac{1}{2}p''_{X_s} + \frac{1}{6}p^{(3)}_{X_s}(x - X_s) + \frac{1}{24}p^{(4)}_{X_s}(x - X_s)^2 + \frac{1}{120}p^{(5)}(x - X_s)^3\,.$$

So

$$\frac{1}{p'_{X_s}} f^2_{xX_s} = \frac{p'_{X_s}}{(x-X_s)^2} + \frac{p''_{X_s}}{x-X_s} + \frac{1}{4}\frac{[p''_{X_s}]^2}{p'_{X_s}} + \frac{1}{3}p^{(3)}_{X_s}$$
$$+ \frac{1}{6}\left(\frac{p''_{X_s}p^{(3)}_{X_s}}{p'_{X_s}} + \frac{1}{2}p^{(4)}_{X_s}\right)(x-X_s)$$
$$+ \frac{1}{12}\left(\frac{1}{3}\frac{[p^{(3)}]^2}{p'_{X_s}} + \frac{1}{2}\frac{p''_{X_s}p^{(4)}_{X_s}}{p'_{X_s}} + \frac{1}{5}p^{(5)}_{X_s}\right)(x-X_s)^2$$
$$+ O((x-X_s)^3)\,.$$

When $n=5$, we have $\vartheta^{[1]} = \vartheta$ ($\vartheta^{[y]}$ is absent). In line (44),

$$\frac{1}{2}\frac{1}{p'_{X_s}} f_{xX_s}\{\vartheta_x + \vartheta_{X_s}\}$$
$$= \frac{\vartheta_{X_s}}{x-X_s}$$
$$+ \frac{1}{2}\frac{p''_{X_s}}{p'_{X_s}}\vartheta_{X_s} + \frac{1}{2}\vartheta'_{X_s}$$
$$+ \left(\frac{1}{6}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\vartheta_{X_s} + \frac{1}{4}\frac{p''_{X_s}}{p'_{X_s}}\vartheta'_{X_s} + \frac{1}{4}\vartheta''_{X_s}\right)(x-X_s)$$
$$+ \left(\frac{1}{24}\frac{p^{(4)}_{X_s}}{p'_{X_s}}\vartheta_{X_s} + \frac{1}{12}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\vartheta'_{X_s} + \frac{1}{8}\frac{p''_{X_s}}{p'_{X_s}}\vartheta''_{X_s} + \frac{1}{12}\vartheta^{(3)}_{X_s}\right)(x-X_s)^2$$
$$+ O((x-X_s)^3)\,.$$

We use eq. (67). In line (45), we have

$$p'_x\, d_{X_s}\left(\frac{p'_x}{p_x}\right) = \left(p'_{X_s} + p''_{X_s}(x-X_s) + \frac{1}{2}p^{(3)}_{X_s}(x-X_s)^2 + \frac{1}{6}p^{(4)}_{X_s}(x-X_s)^3 + \frac{1}{24}p^{(5)}(x-X_s)^4\right)\frac{\xi_s}{(x-X_s)^2}$$
$$= \frac{p'_{X_s}}{(x-X_s)^2} + \frac{p''_{X_s}}{x-X_s} + \frac{1}{2}p^{(3)}_{X_s} + \frac{1}{6}p^{(4)}_{X_s}(x-X_s) + \frac{1}{24}p^{(5)}(x-X_s)^2\,.$$

Moreover,

$$\langle\vartheta_{X_s}\vartheta_x\rangle_r = \langle\psi_{X_s}\rangle + \langle\vartheta_{X_s}\vartheta'_{X_s}\rangle_r(x-X_s) + \frac{1}{2}\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle_r(x-X_s)^2 + O((x-X_s)^3)$$

where by Lemma 10 and eq. (25),

$$2\langle\vartheta_{X_s}\vartheta'_{X_s}\rangle_r = \langle\psi'_{X_s}\rangle = -\frac{c}{480}\left(p'_{X_s}p^{(4)}_{X_s} - 2p''_{X_s}p^{(3)}\right)\langle\mathbf{1}\rangle + \frac{1}{5}p^{(3)}_{X_s}\langle\vartheta_{X_s}\rangle + \frac{1}{10}p''_{X_s}\langle\vartheta'_{X_s}\rangle - \frac{3}{10}p'_{X_s}\langle\vartheta''_{X_s}\rangle\,.$$

Corollary 30. *Let $n=5$. The values of the following integral as a function of X_s:*

$$\frac{2}{p'_{X_s}}\oint\frac{\langle\vartheta_{X_s}\vartheta_x\rangle}{(x-X_s)^{k+1}}\,\frac{dx}{2\pi i}\,.$$

For $k=0$:

$$\frac{c}{20}\left(\frac{1}{3}p^{(3)}_{X_s} + \frac{7}{16}\frac{[p''_{X_s}]^2}{p'_{X_s}}\right)\langle\mathbf{1}\rangle + \frac{9}{10}\frac{p''_{X_s}}{p'_{X_s}}\langle\vartheta_{X_s}\rangle + \frac{3}{10}\langle\vartheta'_{X_s}\rangle \tag{68}$$

For $k = 1$*:*

$$\frac{c}{120}\left(\frac{7}{4}\frac{p''_{X_s}p^{(3)}_{X_s}}{p'_{X_s}} + p^{(4)}_{X_s}\right)\langle\mathbf{1}\rangle + \frac{11}{30}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\langle\vartheta_{X_s}\rangle + \frac{7}{20}\frac{p''_{X_s}}{p'_{X_s}}\langle\vartheta'_{X_s}\rangle + \frac{1}{5}\langle\vartheta''_{X_s}\rangle \tag{69}$$

For $k = 3$*:*

$$\frac{11}{200}\frac{p^{(5)}}{p'_{X_s}}\langle\vartheta_{X_s}\rangle\,. \tag{70}$$

Proof. For $n = 5$, $\vartheta = \vartheta^{[1]}$ ($\vartheta^{[y]}$ is absent). We use eq. (62) for $\langle\vartheta_x\vartheta_{X_s}\rangle^{[1]}$ and Claim 9. (Alternatively, for $k = 0, 1$, the proof follows from the OPE (24) and Example 29.) For $k = 3$, we also need eq. (67) for $\langle\vartheta^{(3)}\rangle$. $\square$

The integral for $k = 2$ is unknown and gives rise to the introduction of the auxiliary function

$$\tilde{B}_s := \oint \frac{\langle\vartheta_x\vartheta_{X_s}\rangle}{(x - X_s)^3}\frac{dx}{2\pi i}\,. \tag{71}$$

$\tilde{B}_s$ depends on $\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle_r$ in the following way:

$$\begin{aligned}\frac{2}{p'_{X_s}}\tilde{B}_s = {} & \frac{c}{192}\left(\frac{1}{3}\frac{[p^{(3)}]^2}{p'_{X_s}} + \frac{1}{2}\frac{p''_{X_s}p^{(4)}_{X_s}}{p'_{X_s}} + \frac{1}{60}p^{(5)}_x\right)\langle\mathbf{1}\rangle \\ & + \left(\frac{1}{24}\frac{p^{(4)}_{X_s}}{p'_{X_s}}\langle\vartheta_{X_s}\rangle + \frac{1}{20}\left(3\frac{p^{(3)}_{X_s}}{p'_{X_s}} + \frac{[p''_{X_s}]^2}{[p'_{X_s}]^2}\right)\langle\vartheta'_{X_s}\rangle + \frac{3}{40}\frac{p''_{X_s}}{p'_{X_s}}\langle\vartheta''_{X_s}\rangle + \frac{1}{60}\langle\vartheta^{(3)}_{X_s}\rangle\right) \\ & + \frac{1}{p'_{X_s}}\partial_x^2|_{x=X_s}\langle\vartheta_{X_s}\vartheta_x\rangle_r\end{aligned}$$

We shall also need the following integral:

Claim 10. *We assume $n = 5$ and the $(2,5)$ minimal model. Let $\langle\vartheta_1\vartheta_2\rangle^{[1]}$ be the Galois-even part of the Galois splitting (61). The value of the integral*

$$\oint \frac{\langle\vartheta_x\vartheta'_{X_s}\rangle^{[1]}}{(x - X_s)^{k+1}}\frac{dx}{2\pi i}$$

for $k = 2$ *is*

$$\begin{aligned}&\left(\frac{c}{9600}p''_{X_s}p^{(5)}_{X_s} + \frac{c}{288}p^{(3)}_{X_s}p^{(4)}_{X_s}\right)\langle\mathbf{1}\rangle \\ &+ \frac{11}{120}p^{(3)}_{X_s}\langle\vartheta''_{X_s}\rangle + \frac{1}{12}p^{(4)}_{X_s}\langle\vartheta'_{X_s}\rangle + \frac{1}{600}p^{(5)}\langle\vartheta_{X_s}\rangle\,.\end{aligned} \tag{72}$$

Proof. Direct computation, using eq. (62). $\square$

Theorem 12. *We assume $n = 5$ and the $(2,5)$ minimal model. Let X_s be a ramification point. Set*

$$\mathcal{D}_s := d_{X_s} - \frac{c}{8}\,\omega_s\,.$$

Let $\tilde{B}_s$ be the auxiliary function $\tilde{B}_s$ given by eq. (71). We have the following complete set of ODEs:

$$
\begin{aligned}
\mathcal{D}_s\langle \mathbf{1}\rangle &= \frac{2\xi_s}{p'_{X_s}}\langle \vartheta_{X_s}\rangle\,, \\
\mathcal{D}_s\langle \vartheta_x\rangle|_{x=X_s} &= \xi_s\left\{-\frac{7c}{480}p'_{X_s}S(p_x)(X_s)\langle \mathbf{1}\rangle + \frac{9}{10}\frac{p''_{X_s}}{p'_{X_s}}\langle \vartheta_{X_s}\rangle - \frac{7}{10}\langle \vartheta'_{X_s}\rangle\right\}\,, \\
\mathcal{D}_s\langle \vartheta'_x\rangle|_{x=X_s} &= \xi_s\left\{\frac{c}{480}\left(7\frac{p''_{X_s}p^{(3)}_{X_s}}{p'_{X_s}} - p^{(4)}_{X_s}\right)\langle \mathbf{1}\rangle + \frac{11}{30}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\langle \vartheta_{X_s}\rangle + \frac{7}{20}\frac{p''_{X_s}}{p'_{X_s}}\langle \vartheta'_{X_s}\rangle - \frac{3}{10}\langle \vartheta''_{X_s}\rangle\right\}\,, \\
\mathcal{D}_s\langle \vartheta''_x\rangle|_{x=X_s} &= \xi_s\left\{\frac{2}{p'_{X_s}}\tilde{B}_s + \frac{7c}{1920}p^{(5)}\langle \mathbf{1}\rangle\right\}\,, \\
\mathcal{D}_s\tilde{B}_s &= \xi_s\left\{\frac{c}{32000}p''_{X_s}p^{(5)}_{X_s} + \frac{c}{960}p^{(3)}_{X_s}p^{(4)}_{X_s}\right\}\langle \mathbf{1}\rangle \\
&\quad + \xi_s\frac{1607}{24000}p^{(5)}_{X_s}\langle \vartheta_{X_s}\rangle + \xi_s\frac{1}{40}p^{(4)}_{X_s}\langle \vartheta'_{X_s}\rangle + \xi_s\left\{\frac{143}{2400}p^{(3)}_{X_s} + \frac{7c}{640}\frac{[p''_{X_s}]^2}{p'_{X_s}}\right\}\langle \vartheta''_{X_s}\rangle \\
&\quad + \frac{9}{10}\frac{p''_{X_s}}{p'_{X_s}}\tilde{B}_s\,.
\end{aligned}
$$

Proof. The ODE for $\langle \mathbf{1}\rangle$ is eq. (54), which holds for any genus. For $n = 5$, $\vartheta = \vartheta^{[1]}$ ($\vartheta^{[y]}$ is absent). For $k \geq 0$, we obtain from the differential equation in Lemma 9 for $N = 1$,

$$
\begin{aligned}
\frac{1}{k!}\left(d_{X_s}\langle \vartheta^{(k)}_x\rangle|_{x=X_s} - \frac{c}{8}\,\omega_s\langle \vartheta^{(k)}_{X_s}\rangle\right) &= 2\frac{\xi_s}{p'_{X_s}}\oint_\gamma \frac{\langle \vartheta_{X_s}\vartheta_{x'}\rangle}{(x'-X_s)^{k+1}}\,\frac{dx'}{2\pi i} \\
&\quad - \xi_s\oint_\gamma \frac{\langle \vartheta_{x'}\rangle}{(x'-X_s)^{k+2}}\,\frac{dx'}{2\pi i} \\
&\quad - \frac{c}{16}\,\xi_s\langle \mathbf{1}\rangle\oint_\gamma \frac{p'_{x'}}{(x'-X_s)^{k+3}}\,\frac{dx'}{2\pi i}\,.
\end{aligned}
$$

In the following, we list the contributions without the factor of ξ_s. Then for $k = 0$, the first line yields (68). The second line yields

$$-\langle \vartheta'_{X_s}\rangle$$

and the third

$$-\frac{c}{32}p^{(3)}_{X_s}\langle \mathbf{1}\rangle\,.$$

For $k = 1$, the first line yields (69). The second line yields

$$-\frac{1}{2}\langle \vartheta''_{X_s}\rangle$$

and the third

$$-\frac{c}{96}p^{(4)}_{X_s}\langle \mathbf{1}\rangle$$

For $k = 2$, the first line yields $\frac{2}{p'_{X_s}}\tilde{B}_s$. The second line yields

$$-\frac{1}{6}\langle \vartheta^{(3)}_{X_s}\rangle = \frac{c}{160}p^{(5)}_{X_s}\langle \mathbf{1}\rangle\,,$$

by eq. (67), and the third

$$-\frac{c}{384}p^{(5)}_{X_s}\langle\mathbf{1}\rangle\,.$$

We address the ODE for $\tilde{B}_s$. Note that by the Galois splitting (61),

$$\langle\vartheta_1\vartheta_{X_s}\rangle = \langle\vartheta_1\vartheta_{X_s}\rangle^{[1]}\,.$$

We have

$$\tilde{B}_s = \oint\frac{1}{(x_1-X_s)^3}\oint\frac{\langle\vartheta_1\vartheta_2\rangle^{[1]}}{x_2-X_s}\frac{dx_2}{2\pi i}\frac{dx_1}{2\pi i}\,,$$

and

$$\frac{\partial}{\partial X_s}\frac{1}{(x-X_s)^k} = \frac{k}{(x-X_s)^{k+1}}\,,$$

so

$$\begin{aligned}\mathcal{D}_s\tilde{B}_s &= 3\xi_s\oint\frac{\langle\vartheta_1\vartheta_{X_s}\rangle}{(x_1-X_s)^4}\frac{dx_1}{2\pi i}\\ &+\xi_s\oint\frac{1}{(x_1-X_s)^3}\oint\frac{\langle\vartheta_1\vartheta_2\rangle^{[1]}}{(x_2-X_s)^2}\frac{dx_2}{2\pi i}\frac{dx_1}{2\pi i}\\ &+\oint\frac{1}{(x_1-X_s)^3}\oint\frac{1}{x_2-X_s}\mathcal{D}_s\langle\vartheta_1\vartheta_2\rangle^{[1]}\frac{dx_2}{2\pi i}\frac{dx_1}{2\pi i}\,.\end{aligned}\tag{73}$$

Here

$$\left(\mathcal{D}_s\langle\vartheta_1\vartheta_2\rangle^{[1]}\right)|_{x_2=X_s} = (\mathcal{D}_s\langle\vartheta_1\vartheta_2\rangle)\,|_{x_2=X_s}\,.$$

Indeed, we have

$$y_2\sim(x_2-X_s)^{1/2}\,,\quad d_{X_s}y_2\sim(x_2-X_s)^{-1/2}\,,$$

so $\mathcal{D}_s\left(y_1y_2\langle\vartheta_1\vartheta_2\rangle^{[y_1y_2]}\right)$ does not contribute to the integral

$$\oint\frac{\mathcal{D}_s\langle\vartheta_1\vartheta_2\rangle}{x_2-X_s}\frac{dx_2}{2\pi i}\,.$$

Thus using the differential equation from Lemma 9 for $N=2$,

$$\mathcal{D}_s\tilde{B}_s = 2\frac{\xi_s}{p'_{X_s}}\oint\frac{1}{(x_1-X_s)^3}\oint\frac{\langle\vartheta_{X_s}\vartheta_2\vartheta_2\rangle}{x_2-X_s}\frac{dx_2}{2\pi i}\frac{dx_1}{2\pi i}\tag{74}$$

$$+2\xi_s\oint\frac{\langle\vartheta_1\vartheta_{X_s}\rangle}{(x_1-X_s)^4}\frac{dx_1}{2\pi i}\tag{75}$$

$$-\frac{c}{16}\xi_s\oint\frac{p'_1}{(x_1-X_s)^5}\oint\frac{\langle\vartheta_2\rangle}{x_2-X_s}\frac{dx_2}{2\pi i}\frac{dx_1}{2\pi i}\tag{76}$$

$$-\frac{c}{16}\xi_s\oint\frac{\langle\vartheta_1\rangle}{(x_1-X_s)^3}\oint\frac{p'_2}{(x_2-X_s)^3}\frac{dx_2}{2\pi i}\frac{dx_1}{2\pi i}\tag{77}$$

(Note that line (73) has dropped out.) We address line (74). By the OPE,

$$\begin{aligned}&\oint\frac{\langle\vartheta_{X_s}\vartheta_2\vartheta_1\rangle}{x_2-X_s}\frac{dx_2}{2\pi i}\\ &\quad=\lim_{x_2\to X_s}\left[\frac{c}{32}f^2_{2X_s}\langle\vartheta_1\rangle+\frac{1}{4}f_{2X_s}(\langle\vartheta_{X_s}\vartheta_1\rangle+\langle\vartheta_2\vartheta_1\rangle)\right]_{\text{order zero in }(x_2-X_s)}+\langle\psi_{X_s}\vartheta_1\rangle\,.\end{aligned}$$

By Example 29 and eq. (25),

$$\frac{2}{p'_{X_s}}\oint\frac{\langle\vartheta_{X_s}\vartheta_2\vartheta_1\rangle}{x_2-X_s}\frac{dx_2}{2\pi i}=\frac{c}{20}\left(\frac{1}{3}p^{(3)}_{X_s}+\frac{7}{16}\frac{[p''_{X_s}]^2}{p'_{X_s}}\right)\langle\vartheta_1\rangle+\frac{9}{10}\frac{p''_{X_s}}{p'_{X_s}}\langle\vartheta_{X_s}\vartheta_1\rangle+\frac{3}{10}\langle\vartheta'_{X_s}\vartheta_1\rangle\,,$$

cf. eq. (68). It follows

$$\begin{aligned}&\frac{2}{p'_{X_s}}\oint\frac{1}{(x_1-X_s)^3}\oint\frac{\langle\vartheta_{X_s}\vartheta_1\vartheta_2\rangle}{x_2-X_s}\frac{dx_2}{2\pi i}\frac{dx_1}{2\pi i}\\&\qquad=\frac{c}{40}\left(\frac{1}{3}p^{(3)}_{X_s}+\frac{7}{16}\frac{[p''_{X_s}]^2}{p'_{X_s}}\right)\langle\vartheta''_{X_s}\rangle+\frac{9}{10}\frac{p''_{X_s}}{p'_{X_s}}\tilde{B}+\frac{3}{10}\oint\frac{\langle\vartheta'_{X_s}\vartheta_1\rangle}{(x_1-X_s)^3}\frac{dx_1}{2\pi i}\end{aligned}$$

where the latter integral is given by eq. (72). Line (75) is given by eq. (70), and gives

$$\frac{11}{200}\xi_s p^{(5)}_{X_s}\langle\vartheta_{X_s}\rangle\,.$$

Line (76) yields

$$-\frac{c}{384}\xi_s p^{(5)}_{X_s}\langle\vartheta_{X_s}\rangle\,.$$

Line (77) yields

$$-\frac{c}{64}\xi_s p^{(3)}_{X_s}\langle\vartheta''_{X_s}\rangle\,.$$

We conclude that

$$\begin{aligned}\mathcal{D}_s\tilde{B}_s&=\xi_s\left\{\frac{c}{32000}p''_{X_s}p^{(5)}_{X_s}+\frac{c}{960}p^{(3)}_{X_s}p^{(4)}_{X_s}\right\}\langle\mathbf{1}\rangle\\&+\xi_s\left(\frac{11}{200}-\frac{c}{384}+\frac{1}{2000}\right)p^{(5)}_{X_s}\langle\vartheta_{X_s}\rangle\\&+\xi_s\frac{1}{40}p^{(4)}_{X_s}\langle\vartheta'_{X_s}\rangle\\&+\xi_s\left\{\left(\frac{c}{120}-\frac{c}{64}+\frac{11}{400}\right)p^{(3)}_{X_s}+\frac{7c}{640}\frac{[p''_{X_s}]^2}{p'_{X_s}}\right\}\langle\vartheta''_{X_s}\rangle\\&+\frac{9}{10}\frac{p''_{X_s}}{p'_{X_s}}\tilde{B}_s\,.\end{aligned}$$

□

10 Comparison with the approach using transcendental methods

We discuss the connection with the work by Mason & Tuite [14].

10.1 The differential equation for the characters of the $(2,5)$ minimal model

The character $\langle\mathbf{1}\rangle$ of any CFT on the torus Σ_1 solves the ODE [6]

$$\frac{d}{d\tau}\langle\mathbf{1}\rangle=\frac{1}{2\pi i}\oint\langle T(z)\rangle\,dz=\frac{1}{2\pi i}\langle\mathbf{T}\rangle\,,$$

where the contour integral is along the real period, and $\oint dz = 1$. It is a particular feature of $g = 1$ that $\langle \mathbf{T} \rangle$ is constant in position. $\langle \mathbf{T} \rangle$ defines modular form of weight two in the modulus. In the $(2,5)$ minimal model, we find

$$2\pi i \frac{d}{d\tau} \langle \mathbf{T} \rangle = \oint \langle T(w)T(z) \rangle \, dz = -4\langle \mathbf{T} \rangle G_2 + \frac{22}{5} G_4 \langle \mathbf{1} \rangle \,.$$

In terms of the *Serre derivative*

$$\mathfrak{D}_\ell := \frac{1}{2\pi i} \frac{d}{d\tau} - \frac{\ell}{12} E_2(\tau) \tag{78}$$

(for weight ℓ), the two first order ODEs combine to give the second order ODE [13, 9]

$$\mathfrak{D}_2 \circ \mathfrak{D}_0 \langle \mathbf{1} \rangle = \frac{11}{3600} \, E_4 \langle \mathbf{1} \rangle \,.$$

The two solutions are the famous Rogers-Ramanujan partition functions [5]

$$\langle \mathbf{1} \rangle_1 = q^{\frac{11}{60}} \sum_{n \geq 0} \frac{q^{n^2+n}}{(q)_n} = q^{\frac{11}{60}} \left(1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + \ldots \right) , \tag{79}$$

$$\langle \mathbf{1} \rangle_2 = q^{-\frac{1}{60}} \sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = q^{-\frac{1}{60}} \left(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \ldots \right) . \tag{80}$$

($q = e^{2\pi i \tau}$) named after the Rogers-Ramanujan identities. The first is given by

$$q^{-\frac{11}{60}} \langle \mathbf{1} \rangle_1 = \prod_{n = \pm 2 \bmod 5} (1 - q^n)^{-1}$$

and provides the generating function for the partition which to a given holomorphic dimension $h \geq 0$ attributes the number of linearly independent holomorphic fields present in the $(2,5)$ minimal model. There is a corresponding Rogers-Ramanujan identity for $q^{\frac{1}{60}} \langle \mathbf{1} \rangle_2$ with a similar combinatorical interpretation, but which involves non-holomorphic fields.

10.2 Introduction of the transcendental coordinates

Let $\omega = \omega_1$, $\omega' = \omega_3 \in \mathbb{C}$ with $\mathrm{Im}\,(\omega/\omega') > 0$ be the two elementary half periods so that $\omega_2 = \omega_1 + \omega_3$ is the midpoint of the fundamental cell. The half periods are the points z with $0 = \partial_z \wp(z|\tau) =: \wp'(z|\tau)$. At these points, the Weierstrass $\wp$-function is invariant under point reflection.

In the finite region, a genus one surface is defined by $y^2 = p_3(x)$ where $p_3(x)$ is a order three polynomial of $x = \wp(z|\tau)$, and $y = \wp'(z|\tau)$. Thus the half periods are the ramification points of the $g = 1$ surface in the finite region. At these points, $x = \wp(z|\tau)$ is invariant under point reflection. This leads us to considering the fundamental cell of the torus modulo point reflection at any fixed half period point. The half periods are all equivalent with that regard, as they differ by full periods only. Considering the fundamental cell modulo point reflection at the chosen half period cuts the cell in two halves. The edge between these two halves is itself cut into two and the two pieces are identified through the reflection at its midpoint.

When we perform a linear fractional transformation, close to a ramification point, the lift to the double cover has two possible values, one on each sheet. We map either of the two points to a corresponding pair of points on the double cover of the other $\mathbb{P}^1_{\mathbb{C}}$, one on each sheet. The ambiguity of the lift disappears as we project down to the second $\mathbb{P}^1_{\mathbb{C}}$. The composition of these maps gives a well-defined map $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. By the Riemann Theorem, all $\mathbb{P}^1_{\mathbb{C}}$s are isomorphic, so the map is an automorphism of $\mathbb{P}^1_{\mathbb{C}}$, thus a linear fractional transformations $x \mapsto \frac{ax+b}{cx+d}$. By fixing the points 0 and ∞, we are left with a scaling factor of x as the only degree of freedom.

Let $z, \hat{z}$ be the coordinates on the two fundamental cells modulo point reflection. We cut away a circle about $z = 0$ and $\hat{z} = 0$ and require

$$z\hat{z} = \varepsilon \tag{81}$$

to identify some small annulus centered at $z = 0$ and at $\hat{z} = 0$, respectively. The copy of $\mathbb{P}^1_{\mathbb{C}}$ covered by the torus defined by the modulus τ respectively $\hat{\tau}$ comes with the natural coordinate

$$\xi = \wp = \wp(z|\tau)\,, \quad \hat{\xi} = \hat{\wp} = \wp(\hat{z}|\hat{\tau})\,, \tag{82}$$

respectively. By the expansion of $\wp(z|\tau)$ about $z = 0$ and by (81), we have on the annulus

$$\xi\hat{\xi} \sim \frac{1}{\varepsilon^2} \tag{83}$$

so $\varepsilon\wp_1 \sim \frac{1}{\varepsilon\wp_2}$, but these are not exact equations. We are glueing here annuli centered at ∞ and zero, respectively, on either $\mathbb{P}^1_{\mathbb{C}}$; the respective center point is excluded from the annulus. The result is topologically a $\mathbb{P}^1_{\mathbb{C}}$, and it is covered by a $g = 2$ surface.

10.3 Pair of almost global coordinates

The new $\mathbb{P}^1_{\mathbb{C}}$ comes with a pair $(X, \hat{X})$ of coordinates satisfying the following properties:

1. X is defined on $\mathbb{P}^1_{\mathbb{C}}$ except for the point (∞) where $\hat{\xi} = 0$, and $\hat{X}$ is defined on $\mathbb{P}^1_{\mathbb{C}}$ except for the point (zero) where $\xi = 0$.
2. We have $X \approx \xi$ where ξ is defined, and $\hat{X} \approx \hat{\xi}$ where $\hat{\xi}$ is defined. On the annulus on which the formerly separate two copies of $\mathbb{P}^1_{\mathbb{C}}$ overlap (and nowhere else), both approximate equations hold simultaneously.
3. The pair $X, \hat{X}$ satifies the exact identity
$$X\hat{X} = \frac{1}{\varepsilon^2}\,, \tag{84}$$
on all of $\mathbb{P}^1_{\mathbb{C}}$.

We shall construct these almost global coordinates. On the annulus, by the approximate eq. (83),

$$\log\xi + \log\hat{\xi} = f(\xi)\,.$$

To this corresponds to the transition rule on the annulus

$$\hat{\xi} = \xi^{-1}e^{f(\xi)}\,. \tag{85}$$

More specifically, we have by eq. (81),

$$\hat{\xi} = \wp\left(\frac{\varepsilon}{\wp^{-1}(\xi|\tau)}\middle|\hat{\tau}\right) .$$

Now the argument goes as follows: $f = \log \xi\hat{\xi} = \log \wp\hat{\wp}$ is nearly constant on the annulus by the fact that $\wp \sim \frac{1}{z^2}$ and by eq. (81). The corrections are small for small ε. Thus f has a Laurent series expansion part of which can be analytically continued to small ξ, and the other part to small $\hat{\xi}$, i.e. to the outside of the annulus (using holomorphicity of f in ε). For X_0 inside the annulus,

$$f(X_0) = \oint_{\text{outer}} \frac{f(X)}{X - X_0} dX - \oint_{\text{inner}} \frac{f(\hat{X})}{\hat{X} - X_0} d\hat{X} .$$

Here by outer resp. inner contour we mean the circle bounding the annulus in the $\hat{\tau}$ and the τ part, respectively. The integral over the outer contour can be extended to the τ part, giving rise to a holomorphic function A, while the integral over the inner contour can be extended to the $\hat{\tau}$ part, giving rise to a holomorphic function $\hat{A}$,

$$f = \log \wp\hat{\wp} = A + \hat{A}$$

It follows that

$$e^f = \xi\hat{\xi} = e^A e^{\hat{A}} ,$$

or

$$\frac{\xi}{e^A}\frac{\hat{\xi}}{e^{\hat{A}}} = 1 .$$

This is the general argument, and we perform the computation for $X, \hat{X}$ explicitely as an expansion in ε.

Claim 11. *Let ξ_1, ξ_2 be given by eqs (82). $\mathbb{P}^1_{\mathbb{C}}$ admits a pair of global coordinates $X = X(\xi, \hat{\xi})$, $\hat{X} = \hat{X}(\xi, \hat{\xi})$ which satisfies eq. (84). In the notations*

$$\hat{z}^2\hat{\wp} = 1 + \sum_{m=1}^{\infty} a_m \hat{z}^{2m+2} ,$$

$$z^2\wp = 1 + \sum_{m=1}^{\infty} \tilde{a}_m z^{2m+2} ,$$

these coordinates are given up to terms of order ε^6, by

$$X = \wp\,\left(1 + a_1\varepsilon^4\left(\wp^2 - 2\tilde{a}_1\right) + a_2\varepsilon^6\left(\wp^3 - 5\tilde{a}_1\wp - 3\tilde{a}_2\right) + O(\varepsilon^8)\right)^{-1} ,$$

$$\hat{X} = \hat{\wp}\,\left(1 + \tilde{a}_1\varepsilon^4\left(\hat{\wp}^2 - 2a_1\right) + \tilde{a}_2\varepsilon^6\left(\hat{\wp}^3 - 5a_1\hat{\wp} - 3a_2\right) + O(\varepsilon^8)\right)^{-1} .$$

Proof. With the notations introduced above, we define

$$\log X := \log \xi - \sum_{n=1}^{\infty} A_n \xi^n ,$$

$$\log \hat{X} := \log \hat{\xi} - \sum_{n=1}^{\infty} B_n \hat{\xi}^n .$$

It follows that

$$X = \frac{\xi}{e^{\sum_{n=1}^{\infty} A_n \xi^n}} \ , \quad \hat{X} = \frac{\hat{\xi}}{e^{\sum_{n=1}^{\infty} B_n \hat{\xi}^n}} \ ,$$

and $\log X + \log \hat{X} = -2 \log \varepsilon$, or

$$X\hat{X} = \frac{1}{\varepsilon^2} \ .$$

Here the coefficients A_n, B_n are determined by the expansion

$$\log(z^2 \xi) + \log(\hat{z}^2 \hat{\xi}) = \sum_{n=1}^{\infty} A_n \xi^n + \sum_{n=1}^{\infty} B_n \hat{\xi}^n \tag{86}$$

on the annulus, and depend both on $\tau, \hat{\tau}$ and ε. The series converge for small enough ε. Details of the computation are left to the reader. □

The closed form of the denominator of X and $\hat{X}$, respectively, defines coefficient matrices which satisfy a system of equations equivalent to that in [14].

10.4 Ramification points using transcendental methods

In the conventions of [14], the $g = 1$ fundamental cell is spanned by $2\omega = 2\pi i$ and $2\omega' = 2\pi i \tau$, (with $\mathrm{Im}\,(1/\tau) = \mathrm{Im}\,(\bar{\tau}) > 0$). The Eisenstein series is

$$E_{2,\tau}^{\mathrm{MT}} = -\frac{1}{12} E_{2,\tau} = -\frac{1}{12} + 2q + \dots \ .$$

The half-periods $\omega_1, \omega_2, \omega_3$ are ω, ω' and $\omega + \omega'$ in some order. Let [1, p. 633]

$$\wp(\omega_k | \tau) = \xi_{k-1} \ , \quad (k = 1, 2, 3) \ .$$

We have

$$[\wp'(z)]^2 = p_3(\wp) = 4 \prod_{k=0}^{2} (\wp(z) - \xi_k) \ .$$

The specific cubic polynomial is given by

$$[\wp']^2 = 4(\wp^3 - 30 G_4 \wp - 70 G_6)$$

and implies that

$$\xi_1 + \xi_2 + \xi_3 = 0 \ . \tag{87}$$

Another natural definition is

$$e_k = -2 \frac{\mathcal{D} \vartheta_k}{\vartheta_k} \ , \quad (k = 2, 3, 4) \tag{88}$$

where $\mathcal{D}$ is the Serre differential operator defined by eq. (78) (the theta functions have weight $1/2$). In the normalisation of Mason and Tuite ($\omega = i\pi$), we have for either torus [1, p. 650]

$$\begin{aligned} e_4 &= \frac{1}{12}(\vartheta_2^4 + \vartheta_3^4) = \xi_1 \\ e_3 &= \frac{1}{12}(-\vartheta_2^4 + \vartheta_4^4) = \xi_0 \\ e_2 &= \frac{1}{12}(-\vartheta_3^4 - \vartheta_4^4) = \xi_2 \ . \end{aligned}$$

Note that by the Jacobi identity (1),

$$\xi_0 - \xi_2 = \frac{1}{4}\vartheta_4^4$$
$$\xi_1 - \xi_0 = \frac{1}{4}\vartheta_2^4$$
$$\xi_1 - \xi_2 = \frac{1}{4}\vartheta_3^4$$

Let the second torus have modulus $\hat{\tau}$ and ramification points $\hat{\xi}_k$. Then the corresponding equations hold for $\hat{\xi}_k$ in terms of the theta functions in $\hat{\tau}$. The ramification points for the $g = 2$ surface obtained by sewing are ξ_0, ξ_1, ξ_2 and, for $k = 0, 1, 2$,

$$\xi_{k+3} = \frac{1}{\varepsilon^2 \hat{\xi}_k} . \tag{89}$$

Claim 12. *Let X_k be the point corresponding to ξ_k by means of Claim 11. The linear rational transformation mapping X_0, X_1, X_2 to $0, 1, \infty$ differs from that mapping ξ_0, ξ_1, ξ_2 to $0, 1, \infty$ only to order at least ε^6. Thus it maps*

$$X_{k+3} = \frac{1}{\varepsilon^2 \hat{X}_k}$$

to

$$f\left(\frac{1}{\varepsilon^2 \hat{X}}\right) = \frac{\vartheta_3^4}{\vartheta_2^4}\left(1 - \frac{\vartheta_4^4}{4}\varepsilon^2 \hat{X}_k - \frac{\vartheta_4^4}{4}\xi_2 \varepsilon^4 \hat{X}_k^2 + O(\varepsilon^6)\right) .$$

Proof. Cf. Appendix H. □

10.5 Ramification points using algebraic methods, for $g = 2$

We set

$$e\{x\} = \exp(2\pi i\, x) .$$

Following [16], we define

$$\theta\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}(\vec{z}, \Omega) = \sum_{\vec{n}\in\mathbb{Z}^g} e\{\frac{1}{2}(\vec{n} + \vec{a})^t \Omega (\vec{n} + \vec{a}) + (\vec{n} + \vec{a})^t (\vec{z} + \vec{b})\} , \quad \forall\, \vec{a}, \vec{b} \in \mathbb{Q}^g .$$

also called the first order theta function with characteristic $\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}$ for $\vec{a}, \vec{b} \in \mathbb{Q}^g$. We assume $g = 2$ and period matrix

$$\Omega = \begin{pmatrix} \Omega_{11} & \nu \\ \nu & \Omega_{22} \end{pmatrix} , \quad \mathrm{Im}\,(\Omega_{jj})\, , \mathrm{Im}\,(\nu) > 0 .$$

In [14], $\Omega_{12} = \Omega_{21} = \nu = O(\varepsilon)$. We adopt the convention

$$\lim_{\nu \to 0} \Omega_{jj} = \tau_j ,$$

where $\tau_1 = \tau$ and $\tau_2 = \hat{\tau}$. To leading order Ω_{jj} and τ_j are the same and their difference lies in $O(\nu^2)$. For terms of order ν^2 and higher, greater care must be taken.

In what follows we take $\vec{z} = \vec{0}$, and if $\vec{a} = (a_1, a_2)^t$ and $\vec{b} = (b_1, b_2)^t$, we write

$$\theta\left[\begin{array}{c}\vec{a}\\ \vec{b}\end{array}\right](0,\Omega) = \theta\left[\begin{array}{c}a_1, a_2\\ b_1, b_2\end{array}\right](\Omega)\,.$$

We set

$$\begin{aligned}
\varrho_j &= e^{2\pi i\,\Omega_{jj}}\,,\\
\lambda &= e^{2\pi i\,\nu} = e^{\tilde{\nu}} = 1 + \tilde{\nu} + \frac{1}{2}\tilde{\nu}^2 + \frac{1}{3!}\tilde{\nu}^3 + \ldots \qquad (\tilde{\nu} = 2\pi i\,\nu)
\end{aligned}$$

So

$$\begin{aligned}
\theta\left[\begin{array}{c}a_1, a_2\\ b_1, b_2\end{array}\right](\Omega) &= \sum_{\vec{n}\in\mathbb{Z}^2} e\{\frac{1}{2}\Omega_{11}(n_1+a_1)^2 + \nu(n_1+a_1)(n_2+a_2) + \frac{1}{2}\Omega_{22}(n_2+a_2)^2\}\; e\{(\vec{n}+\vec{a})^t\vec{b}\}\\
&= \sum_{\vec{n}\in\mathbb{Z}^2} \varrho_1^{\frac{1}{2}(n_1+a_1)^2}\varrho_2^{\frac{1}{2}(n_2+a_2)^2}\lambda^{(n_1+a_1)(n_2+a_2)}e^{2\pi i\{(n_1+a_1)b_1+(n_2+a_2)b_2\}}\,.
\end{aligned}$$

In the following, we assume

$$\vec{a}\cdot\vec{b} = 0\,.$$

Thus

$$\theta\left[\begin{array}{c}a_1, a_2\\ b_1, b_2\end{array}\right](\Omega) = \sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\sum_{k=0}^{\infty}\frac{\tilde{\nu}^k}{k!}(n_1+a_1)^k(n_2+a_2)^k e^{2\pi i n_1 b_1}\varrho_1^{\frac{1}{2}(n_1+a_1)^2}\; e^{2\pi i n_2 b_2}\varrho_2^{\frac{1}{2}(n_2+a_2)^2}\,.$$

Observe that when $a_i = 0$ for at least one $i \in \{1,2\}$ then all summands to odd k drop out. Consider e.g.

$$\theta\left[\begin{array}{c}0, a_2\\ b_1, b_2\end{array}\right](\Omega) = \sum_{n_1\in\mathbb{Z}}\sum_{k=0}^{\infty}\frac{\tilde{\nu}^k}{k!}\; e^{2\pi i n_1 b_1}n_1^k\varrho_1^{\frac{1}{2}n_1^2}\;\sum_{n_2\in\mathbb{Z}}(n_2+a_2)^k e^{2\pi i n_2 b_2}\varrho_2^{\frac{1}{2}(n_2+a_2)^2}\,.$$

Since

$$(\pi i)^k(n_j+a_j)^{2k}\varrho_j^{\frac{1}{2}(n_j+a_1)^2} = \frac{d^k}{d\Omega_{11}^k}\varrho_j^{\frac{1}{2}(n_j+a_1)^2}\,,$$

we find (using the definition of $\tilde{\nu}$)

$$\begin{aligned}
\theta\left[\begin{array}{c}0, a_2\\ b_1, b_2\end{array}\right](\Omega) &= \sum_{n_1\in\mathbb{Z}}\sum_{k=0}^{\infty}\frac{\tilde{\nu}^{2k}}{(2k)!}e^{2\pi i n_1 b_1}n_1^{2k}\varrho_1^{\frac{1}{2}n_1^2}\;\sum_{n_2\in\mathbb{Z}}(n_2+a_2)^{2k}e^{2\pi i n_2 b_2}\varrho_2^{\frac{1}{2}(n_2+a_2)^2}\\
&= \sum_{k=0}^{\infty}\frac{(2\nu)^{2k}}{(2k)!}\sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\frac{d^k}{d\Omega_{11}^k}\left(e^{2\pi i n_1 b_1}\varrho_1^{\frac{1}{2}n_1^2}\right)\frac{d^k}{d\Omega_{22}^k}\left(e^{2\pi i n_2 b_2}\varrho_2^{\frac{1}{2}(n_2+a_2)^2}\right)\,.
\end{aligned}$$

Writing $\vartheta_{k,\Omega_{jj}} = \vartheta_k(0,\varrho_j)$, we obtain

$$\begin{aligned}
\theta\left[\begin{array}{c}\frac{1}{2}\\ 0\end{array}\right](\Omega_{jj}) &= \sum_{n_j\in\mathbb{Z}}\varrho_j^{\frac{1}{2}(n_j+\frac{1}{2})^2} = 2\varrho^{\frac{1}{8}}\sum_{n_j=0}^{\infty}\varrho_j^{\frac{1}{2}n_j(n_j+1)} = \vartheta_{2,\Omega_{jj}}\\
\theta\left[\begin{array}{c}0\\ 0\end{array}\right](\Omega_{jj}) &= \sum_{n_j\in\mathbb{Z}}\varrho_j^{\frac{1}{2}n_j^2} = 1 + 2\sum_{n_j=1}^{\infty}\varrho_j^{\frac{1}{2}n_j^2} = \vartheta_{3,\Omega_{jj}}\\
\theta\left[\begin{array}{c}0\\ \frac{1}{2}\end{array}\right](\Omega_{jj}) &= \sum_{n_j\in\mathbb{Z}}(-1)^{n_j}\varrho_2^{\frac{1}{2}n_j^2} = 1 + 2\sum_{n_j=1}^{\infty}(-1)^{n_j}q^{\frac{1}{2}n_j^2} = \vartheta_{4,\Omega_{jj}}\,.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
\Theta_{3,3} := \theta\left[\begin{array}{c} 0,0 \\ 0,0 \end{array}\right](\Omega) &= \sum_{\vec{n}\in\mathbb{Z}^2} e\{\frac{1}{2}\Omega_{11}n_1^2 + \nu n_1 n_2 + \frac{1}{2}\Omega_{22}n_2^2\} \\
&= \sum_{\vec{n}\in\mathbb{Z}^2} \varrho_1^{\frac{1}{2}n_1^2}\,\varrho_2^{\frac{1}{2}n_2^2}\,\lambda^{n_1 n_2} \\
&= \vartheta_{3,\Omega_{11}}\,\vartheta_{3,\Omega_{22}}\times \\
&\quad\times\left(1 + \frac{(2\nu)^2}{2!}\,\frac{\vartheta'_{3,\Omega_{11}}\,\vartheta'_{3,\Omega_{22}}}{\vartheta_{3,\Omega_{11}}\,\vartheta_{3,\Omega_{22}}} + \frac{(2\nu)^4}{4!}\,\frac{\vartheta''_{3,\Omega_{11}}\,\vartheta''_{3,\Omega_{22}}}{\vartheta_{3,\Omega_{11}}\,\vartheta_{3,\Omega_{22}}} + \frac{(2\nu)^6}{6!}\,\frac{\vartheta^{(3)}_{3,\Omega_{11}}\,\vartheta^{(3)}_{3,\Omega_{22}}}{\vartheta_{3,\Omega_{11}}\,\vartheta_{3,\Omega_{22}}} + O(\nu^8)\right) \\
&= \theta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](\Omega_{11})\,\theta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](\Omega_{22})(1 + O(\nu^2)\,,
\end{aligned}
$$

$$
\begin{aligned}
\Theta_{2,3} := \theta\left[\begin{array}{c} \frac{1}{2},0 \\ 0,0 \end{array}\right](\Omega) &= \sum_{\vec{n}\in\mathbb{Z}^2} e\{\frac{1}{2}\Omega_{11}(n_1+\frac{1}{2})^2 + \nu(n_1+\frac{1}{2})n_2 + \frac{1}{2}\Omega_{22}n_2^2\} \\
&= \sum_{\vec{n}\in\mathbb{Z}^2} \varrho_1^{\frac{1}{2}(n_1+\frac{1}{2})^2}\,\varrho_2^{\frac{1}{2}n_2^2}\,\lambda^{(n_1+\frac{1}{2})n_2} \\
&= \vartheta_{2,\Omega_{11}}\,\vartheta_{3,\Omega_{22}}\times \\
&\quad\times\left(1 + \frac{(2\nu)^2}{2!}\,\frac{\vartheta'_{2,\Omega_{11}}\,\vartheta'_{3,\Omega_{22}}}{\vartheta_{2,\Omega_{11}}\,\vartheta_{3,\Omega_{22}}} + \frac{(2\nu)^4}{4!}\,\frac{\vartheta''_{2,\Omega_{11}}\,\vartheta''_{3,\Omega_{22}}}{\vartheta_{2,\Omega_{11}}\,\vartheta_{3,\Omega_{22}}} + \frac{(2\nu)^6}{6!}\,\frac{\vartheta^{(3)}_{2,\Omega_{11}}\,\vartheta^{(3)}_{3,\Omega_{22}}}{\vartheta_{2,\Omega_{11}}\,\vartheta_{3,\Omega_{22}}} + O(\nu^8)\right)
\end{aligned}
$$

$$
\begin{aligned}
\Theta_{3,2} := \theta\left[\begin{array}{c} 0,\frac{1}{2} \\ 0,0 \end{array}\right](\Omega) &= \sum_{\vec{n}\in\mathbb{Z}^2} e\{\frac{1}{2}\Omega_{11}n_1^2 + \nu n_1(n_2+\frac{1}{2}) + \frac{1}{2}\Omega_{22}(n_2+\frac{1}{2})^2\} \\
&= \sum_{\vec{n}\in\mathbb{Z}^2} \varrho_1^{\frac{1}{2}n_1^2}\,\varrho_2^{\frac{1}{2}(n_2+\frac{1}{2})^2}\,\lambda^{n_1(n_2+\frac{1}{2})} \\
&= \vartheta_{3,\Omega_{11}}\,\vartheta_{2,\Omega_{22}}\times \\
&\quad\times\left(1 + \frac{(2\nu)^2}{2!}\,\frac{\vartheta'_{3,\Omega_{11}}\,\vartheta'_{2,\Omega_{22}}}{\vartheta_{3,\Omega_{11}}\,\vartheta_{2,\Omega_{22}}} + \frac{(2\nu)^4}{4!}\,\frac{\vartheta''_{3,\Omega_{11}}\,\vartheta''_{2,\Omega_{22}}}{\vartheta_{3,\Omega_{11}}\,\vartheta_{2,\Omega_{22}}} + \frac{(2\nu)^6}{6!}\,\frac{\vartheta^{(3)}_{3,\Omega_{11}}\,\vartheta^{(3)}_{2,\Omega_{22}}}{\vartheta_{3,\Omega_{11}}\,\vartheta_{2,\Omega_{22}}} + \dots\right)
\end{aligned}
$$

$$
\begin{aligned}
\Theta_{2,4} := \theta\left[\begin{array}{c} \frac{1}{2},0 \\ 0,\frac{1}{2} \end{array}\right](\Omega) &= \sum_{\vec{n}\in\mathbb{Z}^2} e\{\frac{1}{2}\Omega_{11}(n_1+\frac{1}{2})^2 + \nu(n_1+\frac{1}{2})n_2 + \frac{1}{2}\Omega_{22}n_2^2\}\;e\{\frac{n_2}{2}\} \\
&= \sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}} \varrho_1^{\frac{1}{2}(n_1+\frac{1}{2})^2}\;e^{\pi i n_2}\varrho_2^{\frac{1}{2}n_2^2}\,\lambda^{(n_1+\frac{1}{2})n_2} \\
&= \vartheta_{2,\Omega_{11}}\,\vartheta_{4,\Omega_{22}}\times \\
&\quad\times\left(1 + \frac{(2\nu)^2}{2!}\,\vartheta'_{2,\Omega_{11}}\,\vartheta'_{4,\Omega_{22}} + \frac{(2\nu)^4}{4!}\,\vartheta''_{2,\Omega_{11}}\,\vartheta''_{4,\Omega_{22}} + \frac{(2\nu)^6}{6!}\,\vartheta^{(3)}_{2,\Omega_{11}}\,\vartheta^{(3)}_{4,\Omega_{22}} + \dots\right)
\end{aligned}
$$

$$\Theta_{3,4} := \theta\begin{bmatrix} 0,0 \\ 0,\frac{1}{2} \end{bmatrix}(\Omega) = \sum_{\vec{n}\in\mathbb{Z}^2} e\{\frac{1}{2}\Omega_{11}n_1^2 + \nu n_1 n_2 + \frac{1}{2}\Omega_{22}n_2^2\}\, e\{\frac{n_2}{2}\}$$
$$= \sum_{\vec{n}\in\mathbb{Z}^2} \varrho_1^{\frac{1}{2}n_1^2}\, e^{\pi i n_2} \varrho_2^{\frac{1}{2}n_2^2}\, \lambda^{n_1 n_2}$$
$$= \vartheta_{3,\Omega_{11}}\, \vartheta_{4,\Omega_{22}} \times$$
$$\times\left(1 + \frac{(2\nu)^2}{2!}\, \frac{\vartheta'_{3,\Omega_{11}}\, \vartheta'_{4,\Omega_{22}}}{\vartheta_{3,\Omega_{11}}\, \vartheta_{4,\Omega_{22}}} + \frac{(2\nu)^4}{4!}\, \frac{\vartheta''_{3,\Omega_{11}}\, \vartheta''_{4,\Omega_{22}}}{\vartheta_{3,\Omega_{11}}\, \vartheta_{4,\Omega_{22}}} + \frac{(2\nu)^6}{6!}\, \frac{\vartheta^{(3)}_{3,\Omega_{11}}\, \vartheta^{(3)}_{4,\Omega_{22}}}{\vartheta_{3,\Omega_{11}}\, \vartheta_{4,\Omega_{22}}} + \ldots\right)$$

The following does not fit into this scheme, but a similar argument applies: Here we need $a_i = \frac{1}{2}$ for at least one $i \in \{1,2\}$. For example,

$$\theta\begin{bmatrix} \frac{1}{2}, a_2 \\ 0, b_2 \end{bmatrix}(\Omega) = \sum_{n_1\in\mathbb{Z}} \sum_{k=0}^{\infty} \frac{\tilde{\nu}^k}{k!}(n_1 + \frac{1}{2})^k (n_2 + a_2)^k \varrho_1^{\frac{1}{2}(n_1+\frac{1}{2})^2} \sum_{n_2\in\mathbb{Z}} e^{2\pi i n_2 b_2} \varrho_2^{\frac{1}{2}(n_2+a_2)^2}\,.$$

Since for $n \in \mathbb{Z}$,

$$((n-1) + \frac{1}{2})^k + (-n + \frac{1}{2})^k = ((n-1) + \frac{1}{2})^k + (-1)^k (n - \frac{1}{2})^k$$

vanishes for k odd, we restrict again the summation to even k. We conclude that

$$\Theta_{2,2} := \theta\begin{bmatrix} \frac{1}{2},\frac{1}{2} \\ 0,0 \end{bmatrix}(\Omega) = \sum_{\vec{n}\in\mathbb{Z}^2} e\{\frac{1}{2}\Omega_{11}(n_1 + \frac{1}{2})^2 + \nu(n_1 + \frac{1}{2})(n_2 + \frac{1}{2}) + \frac{1}{2}\Omega_{22}(n_2 + \frac{1}{2})^2\}$$
$$= \sum_{\vec{n}\in\mathbb{Z}^2} \varrho_1^{\frac{1}{2}(n_1+\frac{1}{2})^2}\, \varrho_2^{\frac{1}{2}(n_2+\frac{1}{2})^2}\, \lambda^{(n_1+\frac{1}{2})(n_2+\frac{1}{2})}$$
$$= \sum_{n_1\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}} \sum_{k=0}^{\infty} \frac{\tilde{\nu}^k}{k!}(n_1 + \frac{1}{2})^k (n_2 + \frac{1}{2})^k \varrho_1^{\frac{1}{2}(n_1+\frac{1}{2})^2}\, \varrho_2^{\frac{1}{2}(n_2+\frac{1}{2})^2}$$
$$= \vartheta_{2,\Omega_{11}}\, \vartheta_{2,\Omega_{22}} \times$$
$$\times\left(1 + \frac{(2\nu)^2}{2!}\, \frac{\vartheta'_{2,\Omega_{11}}\, \vartheta'_{2,\Omega_{22}}}{\vartheta_{2,\Omega_{11}}\, \vartheta_{2,\Omega_{22}}} + \frac{(2\nu)^4}{4!}\, \frac{\vartheta''_{2,\Omega_{11}}\, \vartheta''_{2,\Omega_{22}}}{\vartheta_{2,\Omega_{11}}\, \vartheta_{2,\Omega_{22}}} + \frac{(2\nu)^6}{6!}\, \frac{\vartheta^{(3)}_{2,\Omega_{11}}\, \vartheta^{(3)}_{2,\Omega_{22}}}{\vartheta_{2,\Omega_{11}}\, \vartheta_{2,\Omega_{22}}} + O(\nu^8)\right)$$

(The notation $\Theta_{i,j}$ is non-standard.)

In the conventions of [8, see references therein], for $g = 2$, the ramification points are

$$X_0 = 0 = b_5\,, \quad X_1 = 1 = b_6\,, \quad X_2 = b_4\,.$$

Then

$$b_1 = X_3 = \frac{\Theta_{3,3}^2 \Theta_{3,2}^2}{\Theta_{2,3}^2 \Theta_{2,2}^2} = X^{3,3,3,2}_{2,3,2,2}\,,$$
$$b_2 = X_4 = \frac{\Theta_{3,2}^2 \Theta_{3,4}^2}{\Theta_{2,2}^2 \Theta_{2,4}^2} = X^{3,2,3,4}_{2,2,2,4}\,,$$
$$b_3 = X_5 = \frac{\Theta_{3,3}^2 \Theta_{3,4}^2}{\Theta_{2,3}^2 \Theta_{2,4}^2} = X^{3,3,3,4}_{2,3,2,4}\,.$$

Note that we have

$$X^{i,j,k,\ell}_{u,v,s,t} = X^{k,\ell,i,j}_{s,t,u,v}\,. \tag{90}$$

Moreover, we define as in [8]

$$b_0 = \frac{\vartheta^4_{3,\tau_1}}{\vartheta^4_{2,\tau_1}} . \tag{91}$$

We note that when $q = \exp(2\pi i\, \tau)$, we have

$$\frac{\vartheta^4_{2,\tau}}{\vartheta^4_{3,\tau}} = 16q^{\frac{1}{2}}(1 - 8q^{\frac{1}{2}} + 44q - 64q^{\frac{3}{2}} + O(q^2)) \tag{92}$$

The linear fractional transformation that sends X_0 to 0 and X_1 to 1, maps X_2 to b_0. In particular, when $X_0 = 0, X_1 = 1$ then $b_0 = X_2$.

The finite ramification points on the first torus are obtained from X_0, X_1, X_2 in the limit $\nu \to 0$.

Claim 13. *We have*

$$b_0 = \lim_{\nu \to 0} \frac{X_2 - X_0}{X_1 - X_0} = \frac{\xi_2 - \xi_0}{\xi_1 - \xi_0} . \tag{93}$$

In particular, as $\rho_1 \to 0$, $b_0 \to \infty$.

Eq. (93) for the first torus is analogous to eq. (95) for the second torus, which we prove in Claim 15.

Let $X_0, \ldots, X_5$ be the ramification points of the $g = 2$ surface.

Claim 14. *Setting, for $k \geq 0$*

$$R^{(k)}_{i,j} := \frac{\vartheta^{(k)}_{i,\Omega_{11}}}{\vartheta_{i,\Omega_{11}}} \frac{\vartheta^{(k)}_{j,\Omega_{22}}}{\vartheta_{j,\Omega_{22}}} ,$$

where $\vartheta^{(k)}_{i,\Omega_{jj}} = \frac{d^k}{d\Omega^k_{jj}} \vartheta_{i,\Omega_{jj}}$, we have

$$\frac{\Theta^2_{i,j}\Theta^2_{k,\ell}}{\Theta^2_{u,v}\Theta^2_{s,t}} = \frac{\vartheta^2_{i,\Omega_{11}}}{\vartheta^2_{u,\Omega_{11}}} \frac{\vartheta^2_{j,\Omega_{22}}}{\vartheta^2_{v,\Omega_{22}}} \frac{\vartheta^2_{k,\Omega_{11}}}{\vartheta^2_{s,\Omega_{11}}} \frac{\vartheta^2_{\ell,\Omega_{22}}}{\vartheta^2_{t,\Omega_{22}}} R^{i,j,k,\ell}_{u,v,s,t} \tag{94}$$

with

$$
\begin{aligned}
R^{i,j,k,\ell}_{u,v,s,t} = 1 &+ 4v^2\, (R^{(1)}_{i,j} + R^{(1)}_{k,\ell} - R^{(1)}_{u,v} - R^{(1)}_{s,t}) \\
&+ 4v^4 \left(4R^{(1)}_{i,j}R^{(1)}_{k,\ell} + 4R^{(1)}_{u,v}R^{(1)}_{s,t} + \left[R^{(1)}_{i,j}\right]^2 + \left[R^{(1)}_{k,\ell}\right]^2 + 3\left[R^{(1)}_{u,v}\right]^2 + 3\left[R^{(1)}_{s,t}\right]^2\right) \\
&- 16v^4 \left(R^{(1)}_{i,j} + R^{(1)}_{k,\ell}\right)\left(R^{(1)}_{u,v} + R^{(1)}_{s,t}\right) \\
&+ \frac{4}{3}v^4 \left(R^{(2)}_{i,j} + R^{(2)}_{k,\ell} - R^{(2)}_{u,v} - R^{(2)}_{s,t}\right) \\
&+ 16v^6\, (R^{(1)}_{i,j} + R^{(1)}_{k,\ell})\left(4R^{(1)}_{u,v}R^{(1)}_{s,t} + 3\left[R^{(1)}_{u,v}\right]^2 - \frac{1}{3}\,R^{(2)}_{u,v} + 3\left[R^{(1)}_{s,t}\right]^2 - \frac{1}{3}\,R^{(2)}_{s,t}\right) \\
&- 16v^6 \left(R^{(1)}_{u,v} + R^{(1)}_{s,t}\right)\left(4R^{(1)}_{i,j}R^{(1)}_{k,\ell} + \left[R^{(1)}_{i,j}\right]^2 + \frac{1}{3}\,R^{(2)}_{i,j} + \left[R^{(1)}_{k,\ell}\right]^2 + \frac{1}{3}\,R^{(2)}_{k,\ell}\right) \\
&+ 8v^6\Big(2R^{(1)}_{i,j}\left(\left[R^{(1)}_{k,\ell}\right]^2 + \frac{1}{3}\,R^{(2)}_{k,\ell}\right) + 2R^{(1)}_{k,\ell}\left(\left[R^{(1)}_{i,j}\right]^2 + \frac{1}{3}\,R^{(2)}_{i,j}\right) \\
&\qquad + \frac{1}{3}\left(R^{(1)}_{i,j}R^{(2)}_{i,j} + \frac{1}{15}\,R^{(3)}_{i,j}\right) + \frac{1}{3}\left(R^{(1)}_{k,\ell}R^{(2)}_{k,\ell} + \frac{1}{15}\,R^{(3)}_{k,\ell}\right)\Big) \\
&+ 8v^6\Big(-2R^{(1)}_{u,v}\left(3\left[R^{(1)}_{s,t}\right]^2 - \frac{1}{3}\,R^{(2)}_{s,t}\right) - 2R^{(1)}_{s,t}\left(3\left[R^{(1)}_{u,v}\right]^2 - \frac{1}{3}\,R^{(2)}_{u,v}\right) \\
&\qquad + R^{(1)}_{u,v}R^{(2)}_{u,v} - \frac{1}{45}\,R^{(3)}_{u,v} - 4\left[R^{(1)}_{u,v}\right]^3 + R^{(1)}_{s,t}R^{(2)}_{s,t} - \frac{1}{45}\,R^{(3)}_{s,t} - 4\left[R^{(1)}_{s,t}\right]^3\Big) + O(v^8)\,.
\end{aligned}
$$

We have

$$R^{i,j,k,\ell}_{u,v,s,t} = R^{k,\ell,i,j}_{s,t,uv}\,.$$

Proof. Direct calculation. □

In the limit as $v \to 0$, the $g = 2$ surface reduces to a single torus, corresponding to the modulus τ_1. While X_0, X_1, X_2 are the ramification points of the first torus, we find:

Claim 15. *As $v \to 0$,*

$$X_3, X_4, X_5 \;\to\; b_0\,,$$

where b_0 is defined by eq. (91) and Claim 92. We have

$$
\begin{aligned}
X_5 &= b_0(1 + O(v^2)) \\
&= \frac{1}{16q_1^{1/2}}(1 + O(q_1^{1/2}))(1 + O(v^2))\,.
\end{aligned}
$$

Proof. X_3, X_4, X_5 are of the form

$$X^{3,j,3,\ell}_{2,j,2,\ell} = \frac{\vartheta^4_{3,\Omega_{11}}}{\vartheta^4_{2,\Omega_{11}}}R^{3,j,3,\ell}_{2,j,2,\ell}\,, \quad j,\ell \in \{2,3,4\}\,.$$

In particular, for v small,

$$X^{3,j,3,\ell}_{2,j,2,\ell} = b_0(1 + O(v^2))\,.$$

In particular, by eq. (91) and Claim 92

$$X_5 = \frac{1}{16q_1^{1/2}}(1 + O(q_1^{1/2}))(1 + O(v^2))\,.$$

(Recall that Ω_{jj} and τ_j differ by $O(v^2)$ only.) □

Claim 16. *We have*

$$\frac{X_5 - X_3}{X_4 - X_3} = \frac{\vartheta^4_{3,\Omega_{22}}}{\vartheta^4_{2,\Omega_{22}}}(1 + O(\nu^2)) . \tag{95}$$

So when ρ_2 is small, then so is the distance between X_3 and X_4. (Geometrically, the fundamental cell of the torus is stretched to infinity, since as $\rho_2 \to 0$, we have $\Omega_{22} \to \infty$).

Proof. Cf. Appendix I. □

Claim 17. *We have*

$$\frac{X_4 - X_5}{X_3 - X_5} = \left(1 - \frac{\vartheta^4_{2,\Omega_{22}}}{\vartheta^4_{3,\Omega_{22}}}\right)\Big(1 + O(\nu^2)\Big) .$$

Proof. The proof is similar to that of Claim 16. □

Claim 18. *With the conventions of [8], we have*

$$X_3 - X_4 = \frac{\vartheta^4_{3,\Omega_{11}}}{\vartheta^4_{2,\Omega_{11}}}\nu^2\Big(\frac{\pi^2}{4}\vartheta^4_{4,\Omega_{11}}\vartheta^4_{2,\Omega_{22}} + O(\nu^2)\Big) .$$

Moreover,

$$\frac{X_3 - X_5}{X_5} = \frac{\pi^2}{4}\nu^2\vartheta^4_{3,\Omega_{22}}\vartheta^4_{2,\Omega_{11}} + O(\nu^4) .$$

In particular, when ρ_1, ρ_2 are small,

$$\frac{X_3 - X_5}{X_5} \sim \frac{\pi^2}{4}\nu^2 + (1 + O(\nu^4)) .$$

Proof. This calculation is straightforward. □

10.6 Comparison of the $g = 2$ partition functions obtained through either method

Theorem 13. *(Tuite et al.) Let*

$$h_0(q) = \langle \mathbf{1} \rangle_1 , \quad g_0(q) = \langle \mathbf{1} \rangle_2$$

be the $g = 1$ Rogers-Ramanujan partition functions defined by eq. (79) and eq. (80), respectively. In the $(2,5)$ minimal model, the $g = 2$ partition function satisfies a second order PDE whose solutions are, to order ε^2,

$$\begin{aligned}
Z^{(2)}_{V,V}(q_1, q_2, \varepsilon) &= h_0(q_1)h_0(q_2) + O(\varepsilon^2) ,\\
Z^{(2)}_{W,W}(q_1, q_2, \varepsilon) &= g_0(q_1)g_0(q_2) + O(\varepsilon^2) ,\\
Z^{(2)}_{V,W}(q_1, q_2, \varepsilon) &= h_0(q_1)g_0(q_2) + O(\varepsilon^2) ,\\
Z^{(2)}_{W,V}(q_1, q_2, \varepsilon) &= g_0(q_1)h_0(q_2) + O(\varepsilon^2) ,
\end{aligned}$$

where the second order terms are obtained through differentiation of g_0 resp. h_0. In addition, there is a fifth solution given by

$$Z^{(2)}_I(q_1, q_2, \varepsilon) = \varepsilon^{-1/5}\left\{\eta_{\tau_1}^{-2/5}\eta_{\tau_2}^{-2/5} + O(\varepsilon^4)\right\} .$$

Our equations yield the same result up to the expected metric factor [12], and a power of ε which requires a separate argument. Our analysis will be published in the coming weeks. In detail one can write

$$\left(\frac{\vartheta_{2,\tau_i}}{\vartheta_{4,\tau_i}}\right)^2 , \quad i = 1, 2 ,$$

as cross ratio of ramification points and the g_0, h_0 as hypergeometric functions with parameters [12]

$$\begin{aligned} (a, b, c) &= \left(\frac{3}{10}, -\frac{1}{10}, \frac{3}{5}\right) , \\ (a, b, c) &= \left(\frac{7}{10}, \frac{11}{10}, \frac{7}{5}\right) , \end{aligned}$$

respectively.

11 Results to leading order in $X = X_1 - X_2$ only

11.1 Conventions and basic formulae

We shall vary X_s and leave $X_1, \ldots, \widehat{X_s} \ldots, X_n$ fixed. Thus

$$\xi_i = \delta_{is} ,$$

where δ_{ij} is the Kronecker symbol. We have $d = d_{X_s}$ for

$$d_{X_s} := \xi_s \frac{\partial}{\partial X_s} ,$$

and $\omega = \omega_s$. We take

$$X_s = X_1, \quad X := X_1 - X_2$$

and assume X is small.

Definition 31. *By definition, two expressions A, B satisfy*

$$A \cong B$$

if the leading (i.e. lowest order) terms in X in A and B are equal.

For instance,

$$\omega_1 \cong \frac{\xi_1}{X_1 - X_2} = X^{-1}\xi_1 . \tag{96}$$

We shall need the following: Suppose $p_x = a_0 \prod_{i=1}^n (x - X_i)$. We have

$$\begin{aligned} p'_x &= a_0 \sum_{k=1}^n \prod_{i\neq k} (x - X_i) \\ p'_{X_s} &= \frac{d}{dx}|_{x=X_s} p_x = a_0 \prod_{i\neq s} (X_s - X_i) , \\ d_{X_s} p_x &= -\xi_s a_0 \prod_{i\neq s} (x - X_i) \\ (d_{X_s} p)(X_s) &= d_{X_s}|_{x=X_s}\, p_x = -\xi_s p'_{X_s} . \end{aligned} \tag{97}$$

Claim 19. *For $k \geq 1$, we have*

$$\begin{aligned} p^{(k)}_{X_s} &= k \frac{\partial^{k-1}}{\partial X_s^{k-1}} p'_{X_s} \\ d_{X_s} p^{(k)}_{X_s} &= \frac{k}{k+1} \xi_s p^{(k+1)}_{X_s} \end{aligned}$$

and to leading (=lowest) order in $X = X_s - X_2$,

$$p'_{X_s} \mathrel{\tilde{\in}} O(X) , \quad p^{(k)}_{X_s} \mathrel{\tilde{\in}} O(1) \quad \textit{for } k > 1 , \quad d_{X_s} p'_{X_s} \mathrel{\tilde{\in}} O(1) .$$

Proof. (Sketch) For

$$f(x, X_s, X_3, \ldots) = (x - X_s)\, g(x, X_3, \ldots)$$

(where in the following we omit the $X_3, \ldots$, which by assumption are all different from X_s), we have

$$f^{(k)}(X_s, X_s) = kg^{(k-1)}(X_s)\,, \quad k \geq 0\,,$$

since $(x - X_s)$ is linear and vanishes at $x = X_s$. On the other hand,

$$\frac{\partial}{\partial X_s}(f'(X_s, X_s)) = \frac{\partial}{\partial X_s} g(X_s) = g'(X_s)$$

since in g, X_s stands at the place of x. Now to apply these formulae to p, take

$$g(x) = a_0 \prod_{i \neq s}(x - X_i)\,, \quad f_{xX_s} = (x - X_s)a_0 \prod_{i \neq s}(x - X_i) = p_x\,,$$

and observe that

$$g(X_s) = p'_{X_s}\,.$$

In particular, for $X = X_s - X_2$,

$$p^{(k)}_{X_s} = ka_0 g^{(k-1)}(X_s) = k \frac{\partial^{k-1}}{\partial X_s^{k-1}} p'_{X_s}\,,$$

and

$$\begin{aligned} d_{X_s} p'_{X_s} &= d_{X_s} g(X_s) = \xi_s \frac{\partial}{\partial X_s} g(X_s) = \xi_s g'(X_s) = \frac{\xi_s}{2} p''_{X_s} \\ d_{X_s} p''_{X_s} &= 2 d_{X_s} g'(X_s) = 2\xi_s g''(X_s) = \frac{2}{3}\xi_s p^{(3)}_{X_s} \\ d_{X_s} p^{(k)}_{X_s} &= k d_{X_s} g^{(k-1)}(X_s) = k\xi_s g^{(k)}(X_s) = \frac{k}{k+1}\xi_s p^{(k+1)}_{X_s}\,. \end{aligned}$$

□

Moreover,

$$\frac{p''_{X_s}}{p'_{X_s}} \cong 2X^{-1}\,, \tag{98}$$

$$d_{X_s} \frac{1}{p'_{X_s}} = -\xi_s \frac{d_{X_s} p'_{X_s}}{[p'_{X_s}]^2} \cong -\frac{1}{2}\xi_s \frac{p''_{X_s}}{[p'_{X_s}]^2} \cong -\frac{\xi_s}{a_0} X^{-2}\,. \tag{99}$$

Likewise,

$$\frac{p^{(3)}_{X_s}}{p'_{X_s}} = 6X^{-2} + 48X^{-1} \sum_{i \neq s,2} \frac{1}{X_s - X_i} + 6 \left\{ \sum_{i \neq s,2} \frac{1}{(X_s - X_i)^2} + 4 \sum_{\substack{i,j \neq s,2 \\ i \neq j}} \frac{1}{(X_s - X_i)(X_s - X_j)} \right\} \tag{100}$$

The relevant term for us is

$$\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}} \right]_{-1} = 48X^{-1} \sum_{i \neq s,2} \frac{1}{X_s - X_i}\,.$$

Note that $\sum_{i \neq s,2} \frac{1}{X_s - X_i}$ is not a number.

11.2 The first two values for the leading order in the Frobenius ansatz

In the $(2,5)$ minimal model for any genus and to leading (=lowest) order in $X = X_1 - X_2$ only, we have the closed system of ODEs

$$\begin{aligned}\left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle\mathbf{1}\rangle &= \frac{2\xi_s}{p'_{X_s}}\langle\vartheta_{X_s}\rangle\,,\\ \left(d_{X_s} - \frac{c}{8}\omega_s\right)\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}} &\cong 2\xi_s\left[\frac{7c}{640}\left[\frac{p''_{X_s}}{p'_{X_s}}\right]^2\langle\mathbf{1}\rangle + \frac{1}{5}\frac{p''_{X_s}}{p'_{X_s}}\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}}\right]\,.\end{aligned} \tag{101}$$

Because of eq. (96), these ODEs have regular singularities, for which the Frobenius method is available.

Claim 20. *Let $g \geq 1$. Let $u \in \mathbb{R}$ be the leading order of $\langle\mathbf{1}\rangle$ and $\langle\vartheta_{X_s}\rangle$ in the Frobenius ansatz, and let*

$$\bar{u} := u - \frac{c}{8}\,.$$

In the $(2,5)$ minimal model, two values of $\bar{u}$ are given by

$$\frac{11}{10}\,,\quad \frac{7}{10}\,.$$

Proof. We have a reason to assume that $\langle\mathbf{1}\rangle$ and $\langle\vartheta_{X_s}\rangle$ are of the same leading order, and use eq. (97). So the Frobenius ansatz reads

$$\begin{aligned}\langle\mathbf{1}\rangle &\cong a(X_s - X_2)^u\\ \frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}} &\cong b(X_s - X_2)^{u-1}\,,\quad u \in \mathbb{R}\,,\end{aligned}$$

where a, b do not depend on X_s. Thus for $\bar{u} = u - \frac{c}{8}$, this yields

$$\begin{aligned}\bar{u}a &= 2b\,,\\ (\bar{u} - 1)b &= \frac{7c}{80}a + \frac{4}{5}b \quad\Leftrightarrow\quad \left(\bar{u} - \frac{9}{5}\right)b = \frac{7c}{80}a\,,\end{aligned}$$

It follows that

$$\bar{u}\left(\bar{u} - \frac{9}{5}\right) = \frac{7c}{40}\,. \tag{102}$$

In the $(2,5)$ minimal model, $c = -\frac{22}{5}$, so

$$\bar{u}_{1/2} = \frac{9}{10} \pm \sqrt{\frac{81}{100} - \frac{77}{100}} = \frac{9}{10} \pm \frac{1}{5} = \begin{cases}\frac{11}{10} & \text{for } +\\ \frac{7}{10} & \text{for } -\end{cases}\,.$$

$\square$

Since $\frac{c}{8} = -\frac{11}{20}$, it follows that

$$u = \bar{u} - \frac{11}{20} = \begin{cases}\frac{11}{20} & \text{for } +\\ \frac{3}{20} & \text{for } -\end{cases} \tag{103}$$

Thus instead of considering the differential eq. for $\langle\vartheta_x\rangle$, we specialise to that for $\langle\vartheta_{X_s}\rangle$. Since $\langle\vartheta_x\rangle = \langle\vartheta_x\rangle_r$ is a polynomial, only finitely many equations are to be established.

11.3 The ODE for $\langle(\vartheta^{[1]})^{(k)}_{X_s}\rangle$ and $\langle(\vartheta^{[y]})^{(k)}_{X_s}\rangle$

In Subsection 11.2, we have established the differential eqs (54) and (55) for the 0-and 1-point function of ϑ for arbitrary genus, and two values of $\bar{u}$. We shall now restrict to the $(2,5)$ minimal model and establish the third differential equation and the third value for $\bar{u}$.

Claim 21. *(The third value) Let $g \geq 1$ and $k \geq 0$. We assume the $(2,5)$ minimal model.*

1. *To lowest order in $X = X_s - X_2$, we have*

$$\boxed{\left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle(\vartheta^{[1]})^{(k)}_{X_s}\rangle \cong \frac{2\xi_s}{p'_{X_s}}\left[\langle\vartheta_{X_s}(\vartheta^{[1]})^{(k)}_x\rangle\right]_{reg.}}\,. \tag{104}$$

and

$$\boxed{\left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle(\vartheta^{[y]})^{(k)}_{X_s}\rangle \cong \frac{2\xi_s}{p'_{X_s}}\left[\langle\vartheta_{X_s}(\vartheta^{[y]})^{(k)}_x\rangle\right]_{reg.}}\,. \tag{105}$$

2. *In particular, for $k = 1$,*

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\frac{\langle(\vartheta^{[1]})'_{X_s}\rangle}{p'_{X_s}} \cong \xi_s\left[\frac{7c}{480}\frac{p''_{X_s}}{p'_{X_s}}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\langle\mathbf{1}\rangle + \frac{11}{30}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}} - \frac{3}{20}\frac{p''_{X_s}}{p'_{X_s}}\frac{\langle(\vartheta^{[1]})'_{X_s}\rangle}{p'_{X_s}}\right]\,, \tag{106}$$

3. *The third value for $\bar{u} = u - \frac{c}{8}$ is*

$$\frac{7}{10}\,.$$

In order to establish the corresponding differential eq. for $\langle\vartheta''_{X_s}\rangle$, we need to take the terms $\propto (x - X_s)^2$ into consideration. Comparison with $N_2(T,T)$ in the ordinary OPE of T for the $(2,5)$ minimal model does not lead us any further since the space of fields of dimension 6 is two- (rather than one-) dimensional.

Proof. 1. Only contributions from $\frac{2}{p'_{X_s}}\left[\langle\vartheta_{X_s}\vartheta^{[1]}_x\rangle\right]_{\text{no pole}}$ (resp. $\frac{2\xi_s}{p'_{X_s}}\left[\langle\vartheta_{X_s}\vartheta^{[y]}_x\rangle\right]_{\text{no pole}}$) contribute to leading (lowest) order in $X = X_s - X_2$ to the differential equation in Lemma 9. Replacing $\vartheta^{[1]}_x$ (resp. $\vartheta^{[y]}_x$) by its Taylor expansion about $x = X_s$ on both sides of the equation for $\vartheta = \vartheta^{[1]}$ (resp. for $\vartheta = \vartheta^{[y]}$) and comparing the respective coefficient of $(x_2 - X_s)^k$ yields the claimed differential eq. for $\langle\vartheta^{(k)}_x\rangle$ (resp. $\langle\vartheta^{[y]}_x\rangle$).

More specifically, by the Frobenius method,

$$\begin{aligned}\langle\vartheta(x_2)\ldots\rangle &= (X_1 - X_2)^u(a + O(X_1 - X_2))\\ &\cong a(X_1 - X_2)^u\,,\end{aligned}$$

where a is in general a function of $X_2, \ldots, X_n$ and x_2, and

$$\langle\vartheta^{(k)}(X_2)\ldots\rangle = (X_1 - X_2)^u\left(\frac{\partial^k}{dx_2^k}|_{x_2=X_2}a + O(X_1 - X_2)\right)\,.$$

2. In order to actually compute the r.h.s. of eq. (104) for $k = 1$, we use that

$$\langle \vartheta_{X_s} (\vartheta^{[1]})^{(k)}_x \rangle = \frac{\partial^k}{\partial x^k} \langle \vartheta_{X_s} \vartheta^{[1]}_x \rangle \quad \text{for } k \geq 0 \,, \tag{107}$$

and

$$\Big[\frac{\partial^k}{\partial x^k} \langle \vartheta_{X_s} \vartheta^{[1]}_x \rangle \Big]_{\text{reg.}} = \frac{\partial^k}{\partial x^k} \Big[\langle \vartheta_{X_s} \vartheta^{[1]}_x \rangle \Big]_{\text{reg.}} .$$

The splitting of ϑ induces a splitting

$$\langle \vartheta_{X_s} \vartheta_x \rangle = \langle \vartheta_{X_s} \vartheta^{[1]}_x \rangle + y \langle \vartheta_{X_s} \vartheta^{[y]}_x \rangle \,. \tag{108}$$

Here by the graphical representation of $\langle \vartheta_{X_s} \vartheta_x \rangle$, eq. (38), we have (40) and (41). Since we aim at a differential eq. to leading order terms only and since $\frac{d}{dx} p'_{X_s} = 0$, we can immediately restrict our consideration to leading order terms in $\left[\frac{2}{p'_{X_s}} \langle \vartheta_{X_s} \vartheta^{[1]}_x \rangle \right]_{\text{reg.}}$. Using eqs (122) and (124) in the proof of Claim 6,

$$\begin{aligned}
&\frac{2}{p'_{X_s}} \frac{\partial}{\partial x} \left[\frac{c}{32} f^2_{xX_s} \langle \mathbf{1} \rangle + \frac{1}{4} f_{xX_s} \left\{ \langle \vartheta_{X_s} \rangle + \langle \vartheta^{[1]}_x \rangle \right\} \right]_{\text{reg.}} \\
&\cong \frac{c}{96} \frac{p''_{X_s}}{p'_{X_s}} p^{(3)}_{X_s} \langle \mathbf{1} \rangle + \frac{1}{6} \left(\frac{p^{(3)}_{X_s}}{p'_{X_s}} \langle \vartheta_{X_s} \rangle + \frac{3}{2} \frac{p''_{X_s}}{p'_{X_s}} \langle (\vartheta^{[1]})'_{X_s} \rangle \right) \\
&+ \frac{c}{96} \left(\frac{1}{2} \frac{p''_{X_s}}{p'_{X_s}} p^{(4)}_{X_s} + \frac{1}{3} \frac{[p^{(3)}_{X_s}]^2}{p'_{X_s}} \right) \langle \mathbf{1} \rangle (x - X_s) \\
&\qquad + \frac{1}{12} \left(\frac{p^{(4)}_{X_s}}{p'_{X_s}} \langle \vartheta_{X_s} \rangle + 2 \frac{p^{(3)}_{X_s}}{p'_{X_s}} \langle (\vartheta^{[1]})'_{X_s} \rangle + 3 \frac{p''_{X_s}}{p'_{X_s}} \langle (\vartheta^{[1]})''_{X_s} \rangle \right) (x - X_s) \\
&+ O((x - X_s)^2) \,.
\end{aligned} \tag{109}$$

Moreover,

$$\frac{\partial}{\partial x} \langle \vartheta_{X_s} \vartheta^{[1]}_x \rangle_r = \frac{1}{2} \langle \psi'_x \rangle + \langle \vartheta'_x \vartheta'_x \rangle_r \, (X_s - x) + O\Big((X_s - x)^2 \Big) \,, \tag{110}$$

where $\langle \psi_x \rangle = \langle \psi^{[1]}_x \rangle + y \langle \psi^{[y]}_x \rangle$ and

$$\begin{aligned}
\langle \psi'_x \rangle &= \langle (\psi^{[1]})'_x \rangle + y \left(\partial_x + \frac{1}{2} \frac{p'_x}{p_x} \right) \langle \psi^{[y]}_x \rangle \\
&= \partial_x \langle \vartheta^{[1]}_x \vartheta^{[1]}_x \rangle_r + p'_x \langle \vartheta^{[y]}_x \vartheta^{[y]}_x \rangle_r + p_x \partial_x \langle \vartheta^{[y]}_x \vartheta^{[y]}_x \rangle_r + O(y) \,.
\end{aligned}$$

It follows that

$$\frac{\partial}{\partial x} |_{X_s} \langle \vartheta_{X_s} \vartheta^{[1]}_x \rangle_r = \frac{1}{2} \partial_x |_{X_s} \langle \vartheta^{[1]}_x \vartheta^{[1]}_x \rangle_r + \frac{1}{2} p'_{X_s} \langle \vartheta^{[y]}_{X_s} \vartheta^{[y]}_{X_s} \rangle_r \,. \tag{111}$$

To obtain the first term on the r.h.s. of eq. (111) , we differentiate $\langle \psi_x \rangle$ given eq. (25),

$$\begin{aligned}
\partial_x \langle \vartheta^{[1]}_x \vartheta^{[1]}_x \rangle_r = &- \frac{c}{480} \left(p'_x p^{(4)}_x - 2 p''_x p^{(3)}_x \right) \langle \mathbf{1} \rangle \\
&+ \left\{ -\frac{1}{5} p_x \, \partial^3_x - \frac{3}{10} p'_x \, \partial^2_x + \frac{1}{10} p''_x \, \partial_x + \frac{1}{5} p^{(3)}_x \right\} \langle \vartheta^{[1]} \rangle \,.
\end{aligned} \tag{112}$$

$\langle\psi'_x\rangle$ is regular at $x = X_s$, and its derivative at X_s equals

$$\partial_x|_{X_s}\langle\vartheta_x^{[1]}\vartheta_x^{[1]}\rangle_r = -\frac{c}{480}\left(p'_{X_s}p^{(4)}_{X_s} - 2p''_{X_s}p^{(3)}_{X_s}\right)\langle\mathbf{1}\rangle + \left\{-\frac{3}{10}p'_{X_s}\,\partial_x^2|_{X_s} + \frac{1}{10}p''_{X_s}\,\partial_x|_{X_s} + \frac{1}{5}p^{(3)}_{X_s}\right\}\langle\vartheta^{[1]}\rangle\,.$$

The second term on the r.h.s. of of eq. (111) does not contribute to leading order. Multiplying eq. (110) by $\frac{2}{p'_{X_s}}$ and adding to eq. (109) yields $\xi_s^{-1}\left[\langle\vartheta_{X_s}(\vartheta^{[1]})'_x\rangle\right]_{\text{reg.}}$. Evaluated at $x = X_s$, this yields according to eq. (104),

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle(\vartheta^{[1]})'_{X_s}\rangle \cong \xi_s\left[\frac{7c}{480}\frac{p''_{X_s}}{p'_{X_s}}p^{(3)}_{X_s}\langle\mathbf{1}\rangle + \frac{11}{30}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\langle\vartheta_{X_s}\rangle + \frac{7}{20}\frac{p''_{X_s}}{p'_{X_s}}\langle(\vartheta^{[1]})'_{X_s}\rangle\right].$$

(Note that since this is an equation to leading order only, we have omitted terms $\propto p'_{X_s}$.) Using eqs (98) and (99) yields the claimed differential eq. for $\langle(\vartheta^{[1]})'_{X_s}\rangle$.

3. After change to the basis $\langle\mathbf{1}\rangle, \frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}}, \frac{\langle(\vartheta^{[1]})'_{X_s}\rangle}{p'_{X_s}}$, we have

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle\mathbf{1}\rangle = 2\xi_s\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}}\,,$$

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}} \cong \xi_s\left[\frac{7c}{80}X^{-2}\langle\mathbf{1}\rangle + \frac{4}{5}X^{-1}\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}}\right]$$

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\frac{\langle(\vartheta^{[1]})'_{X_s}\rangle}{p'_{X_s}} \cong \xi_s\left[\frac{7c}{240}X^{-1}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\langle\mathbf{1}\rangle + \frac{11}{30}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}} - \frac{3}{10}X^{-1}\frac{\langle(\vartheta^{[1]})'_{X_s}\rangle}{p'_{X_s}}\right],$$

or

$$\begin{pmatrix} \bar{u} & -2 & 0 \\ -\frac{7c}{80} & \bar{u} - \frac{9}{5} & 0 \\ -\frac{7c}{240}\frac{p^{(3)}_{X_s}}{p'_{X_s}} & -\frac{11}{30}\frac{p^{(3)}_{X_s}}{p'_{X_s}} & \bar{u} - \frac{7}{10} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\,,$$

and

$$0 = \det = \left(\bar{u} - \frac{7}{10}\right)\det\begin{pmatrix} \bar{u} & -2 \\ -\frac{7c}{40} & \bar{u} - \frac{9}{5} \end{pmatrix}.$$

So the third value is $\bar{u} = \frac{7}{10}$.

□

11.4 Check: The differential equation for N-point functions of ϑ and its kth derviative, for arbitray genus

We check that no logarithmic solutions can arise in the system.

Lemma 14. *(Differential eq. for the N-point function)*
Let

$$dX_i = \xi_i \quad \textit{with} \quad \xi_1 \neq 0\,, \quad \xi_i = 0 \quad \textit{for } i \neq 1\,.$$

Let $k \geq 0$. We have

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle(\vartheta^{[1]})^{(k)}_{X_s}\ldots\rangle \cong \frac{2\xi_s}{p'_{X_s}}\left[\langle\vartheta_{X_s}(\vartheta^{[1]})^{(k)}_x\ldots\rangle\right]_{\substack{reg.\\ x=X_s}} ,$$

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\frac{\left[\langle\vartheta_{X_s}(\vartheta^{[1]})^{(k)}_x\ldots\rangle\right]_{\substack{reg.\\ x=X_s}}}{p'_{X_s}} \cong 2\xi_s\left[\frac{7c}{640}\left[\frac{p''_{X_s}}{p'_{X_s}}\right]^2\langle(\vartheta^{[1]})^{(k)}_{X_s}\ldots\rangle + \frac{1}{5}\frac{p''_{X_s}}{p'_{X_s}}\frac{\left[\langle\vartheta_{X_s}(\vartheta^{[1]})^{(k)}_x\ldots\rangle\right]_{\substack{reg.\\ x=X_s}}}{p'_{X_s}}\right],$$

where $[\ ]_{reg.}$ denotes the restriction to the terms regular in the first two positions. The system closes up, and setting

$$\left[\langle(\vartheta^{[1]})^{(k)}_{X_s}\ldots\rangle\right]_{reg.} \cong aX^u$$

$$\frac{\left[\langle\vartheta_{X_s}(\vartheta^{[1]})^{(k)}_x\ldots\rangle\right]_{\substack{reg.\\ x=X_s}}}{p'_{X_s}} \cong bX^{u-1}, \quad u \in \mathbb{R},$$

the two values for $\bar{u} = u - \frac{c}{8}$ are

$$\frac{11}{10}, \quad \frac{7}{10}.$$

Remark 32. *We don't know what $\left[\langle\vartheta_{X_s}\vartheta^{(k)}_x\rangle\right]_{\substack{reg.\\ x=X_s}}$ is in general. However, we can conclude (for $k = 2$) that*

$$\bar{u}_{4/5} = \begin{cases}\frac{11}{10}\\ \frac{7}{10}\end{cases}.$$

For $k = 3$, we have an explicit expression for $n = 5$.

Proof. In the following, let

$$\vartheta = \vartheta^{[1]}.$$

Let $X_2, \ldots, X_n$ be fixed and $x_2, \ldots$ are arbitrary, but mutually different and different from X_s. By Lemma 9 and Remark 25, we have

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle\vartheta(x_1)\ldots\rangle \cong \frac{2\xi_s}{p'_{X_s}}\left[\langle\vartheta_{X_s}\vartheta(x_1)\ldots\rangle\right]_{\text{reg.}}, \tag{113}$$

to leading order in X. (On the r.h.s. we have restricted to terms which are regular at $x_2 = X_s$.) Here

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle\vartheta(x_1)\vartheta(x_2)\ldots\rangle|_{x_1=X_s} \cong \left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle\vartheta_{X_s}\vartheta(x_2)\ldots\rangle,$$

so we can replace x_1 by X_s on both sides,

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle\vartheta_{X_s}\ldots\rangle \cong \frac{2\xi_s}{p'_{X_s}}\left[\langle\vartheta_{X_s}\vartheta_x\ldots\rangle\right]_{\substack{\text{reg.}\\ x=X_s}},$$

yielding the first of the claimed equations for $k = 0$. We address the second equation. The same arguments that prove eq. (101) also show

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\frac{\langle\vartheta_{X_s}\vartheta(x_2)\ldots\rangle}{p'_{X_s}} \cong 2\xi_s\left[\frac{7c}{640}\left[\frac{p''_{X_s}}{p'_{X_s}}\right]^2\langle\vartheta(x_2)\ldots\rangle + \frac{1}{5}\frac{p''_{X_s}}{p'_{X_s}}\frac{\langle\vartheta_{X_s}\vartheta(x_2)\ldots\rangle}{p'_{X_s}}\right].$$

We restrict to the terms regular at $x_2 = X_s$. Since by holomorphy of $\langle\vartheta_{X_s}\vartheta(x_2)\rangle$ outside $x_2 = X_s$, the coefficients of its Laurent series expansion can be defined by contour integrals, we have

$$\left[\left(d_{X_s} - \frac{c}{8}\omega_s\right)\langle\vartheta_{X_s}\vartheta(x_2)\ldots\rangle\right]_{\text{reg.}} = \left(d_{X_s} - \frac{c}{8}\omega_s\right)[\langle\vartheta_{X_s}\vartheta(x_2)\ldots\rangle]_{\text{reg.}} \,. \tag{114}$$

Now setting $x_2 = X_s$ yields the second claimed equation for $k = 0$. Alternatively, we replace of $\vartheta(x_2)$ by its Taylor series expansion about $x_2 = X_s$. Comparing the terms $\propto (x_2 - X_s)^k$ yields the claimed system. This system closes up. For the given Frobenius ansatz, the arguments used in the proof of Claim 20, we obtain the two claimed values for $\bar{u}$. □

11.5 The number of equations to leading order

We have

$$d_{X_s}\langle\mathbf{1}\rangle \sim \langle\vartheta^{[1]}_{X_s}\rangle \,.$$

and all differential equations for N-point functions of $\vartheta^{[1]}$ and its derivatives do not involve $\vartheta^{[y]}$. So set $\vartheta = \vartheta^{[1]}$, and let $N \geq 1$. By Lemma 14, for $k = 0, \ldots, n-3$, (with $\sharp\{k\} = \deg\langle\vartheta\rangle$),

$$d_{X_s}\langle\vartheta^{(k)}_{X_s}\rangle \sim \left[\langle\vartheta_{X_s}\vartheta^{(k)}_x\rangle\right]_{\substack{\text{reg.}\\ x-X_s}} \,.$$

In the $(2,5)$ minimal model, the r.h.s. is known for both $k = 0, 1$. For the remaining $n-4$ values of k we have by Lemma 14,

$$d_{X_s}\left[\langle\vartheta_{X_s}\vartheta^{(k)}_x\rangle\right]_{\substack{\text{reg.}\\ x-X_s}} \sim \langle\vartheta^{(k)}\rangle + \left[\langle\vartheta_{X_s}\vartheta^{(k)}_x\rangle\right]_{\substack{\text{reg.}\\ x-X_s}} \,.$$

So

$$d^2_{X_s}\langle\vartheta^{(k)}\rangle \;\sim\; d_{X_s}\langle\vartheta^{(k)}\rangle + \langle\vartheta^{(k)}\rangle \,,$$

and both $\langle\vartheta^{(k)}\rangle$ and $\left[\langle\vartheta_{X_s}\vartheta^{(k)}_x\rangle\right]_{\substack{\text{reg.}\\ x-X_s}}$ are known as functions of X. So to leading order in X, $\langle\mathbf{1}\rangle$ is determined by

$$1 + (n-2) + (n-4) = 2n - 5$$

equations, whenever $n \geq 4$, and $n - 1$ otherwise.

12 Application to the $(2,5)$ minimal model for $g = 2$

12.1 The fifth equation

We need to know $\langle\vartheta_x\rangle$ and $\langle\vartheta_x\vartheta_{X_s}\rangle$ in the $(2,5)$ minimal model for $n = 5$.

1. In the limit of $\langle\vartheta_1\vartheta_2\rangle_r$ as $x_1 \to x_2$,

$$B_0(x_1^2 + x_2^2) + \mathbf{B_{1,1}}x_1x_2 \;\mapsto\; (2B_0 + \mathbf{B_{1,1}})x^2$$

   so knowledge of $\langle\vartheta^2\rangle_r$ determines $\langle\vartheta_1\vartheta_2\rangle_r$ only up to one unknown.

2. We computed $\langle\vartheta_2^2\vartheta_3\rangle_r$ with the $(2,5)$ minimal model property (the formula for ψ_2) implemented. $\langle\vartheta_2^2\vartheta_3\rangle_r$ is a function of $\langle\mathbf{1}\rangle, \langle\vartheta\rangle$ and of $\langle\vartheta_2\vartheta_3\rangle_r$. We considered the change in $\langle\vartheta_2^2\vartheta_3\rangle_r$ produced by

$$\langle\vartheta_i\vartheta_j\rangle_r \;\mapsto\; \langle\vartheta_i\vartheta_j\rangle_r + (x_i - x_j)^2\,, \tag{115}$$

   for $(i,j) = (2,3)$. (Since all terms of order $O(x^3)$ are known in $\langle\vartheta_1\vartheta_2\rangle_r$, this is the only change to consider.) The new terms in $\langle\vartheta_2^2\vartheta_3\rangle_r$ resulting from (115) all lift, along the projection $x_1 \mapsto x_2$, to symmetric polynomials $[k\ell m]$ (i.e. x_1^k, x_2^ℓ, x_3^m+ permutations) of order $k + \ell + m = 5$, namely $[500], [320], [311], [221]$, and $y_1y_2 + y_2y_3 + y_3y_1$ ($[410]$ does not occur). Projecting $[221]$ yields

$$x_1^2x_2^2x_3 + x_1x_2^2x_3^2 + x_1^2x_2x_3^2 \overset{x_1 \to x_2}{\longrightarrow} \underset{\text{with known coeff. } B_{0,0,1}}{x_2^4x_3 + 2x_2^3x_3^2} \sim \underset{\substack{\uparrow \\ \ldots(x_2^2+x_3^2+x_2x_3)}}{\vartheta_2^h(2x_3^2 + x_2x_3)} \overset{x_2 \to x_3}{\longrightarrow} \underset{3B_{0,0,1}\propto 2B_0+\mathbf{B_{1,1}}}{3\vartheta_3^hx_3^2}\,.$$

3. $\langle\vartheta_1\vartheta_2\vartheta_3\rangle_r$ is a function of $\langle\mathbf{1}\rangle, \langle\vartheta\rangle$ and $\langle\vartheta_i\vartheta_j\rangle_r$. We computed $\langle\vartheta_1\vartheta_2\vartheta_3\rangle_r^h = \langle\vartheta_1\vartheta_2\rangle_r\vartheta_3^h$ with $\vartheta_3^h = -\frac{3ca_0}{4}x_3^3.1$ (Laurent series as $x_3 \to \infty$) and the terms produced by the change (115) in $\langle\vartheta_1\vartheta_2\vartheta_3\rangle_r^h$ for $i,j = 1,2,3$. These are $[500], [320], [311]$, which are known as they are of order ≥ 3 in one variable, and $y_1y_2 + y_2y_3 + y_3y_1$. $[221]$ is not produced due to our restriction to $\langle\vartheta_1\vartheta_2\vartheta_3\rangle_r^h$. Unexpectedly, the coefficients of all occurring terms perfectly match, yielding no constraint on $[221]$.

The $(2,5)$ minimal model constraint does not provide any further information on the 3-point function.

For $n = 5$, we are interested in $\mathbf{B_{1,1}}$, so it suffices to formulate the fifth differential eq. for $\langle\vartheta_3\vartheta_3''\rangle_r$ at $x_3 = X_s$, or for $[\langle\vartheta_{X_s}\vartheta_x''\rangle]_{\substack{\text{reg.}\\x=X_s}}$. By Lemma 14, we have

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\frac{[\langle\vartheta_{X_s}\vartheta_x''\rangle]_{\substack{\text{reg.}\\x=X_s}}}{p'_{X_s}} \cong 2\xi_s\left[\frac{7c}{640}\frac{[p''_{X_s}]^2}{p'_{X_s}}\frac{\langle\vartheta''_{X_s}\rangle}{p'_{X_s}} + \frac{1}{5}\frac{p''_{X_s}}{p'_{X_s}}\frac{\left[\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle\right]_{\substack{\text{reg.}\\x=X_s}}}{p'_{X_s}}\right]. \tag{116}$$

Remark 33. *From the formula for $\langle\vartheta_2\vartheta_2\vartheta_3\rangle_r$ we can deduce*

$$\langle\vartheta_1\vartheta_2\vartheta_3\rangle_r \quad \text{mod } \textit{terms in } \ker_{1\to 2}$$

where the kernel is of the form

$$(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2 \times \textit{polynomial}\,.$$

However, the latter is of order $O(x^4)$ and thus known.

12.2 The full matrix of the system of differential equations for $\langle 1\rangle$ and derivatives of $\langle\vartheta\rangle$ for $n = 5$

For $n = 5$, $\vartheta_x^{[y]}$ is absent for degree reason, so

$$\vartheta_x = \vartheta_x^{[1]} \, .$$

Theorem 15. *We assume the* $(2,5)$ *minimal model for* $g = 2$ *(*$n = 5$*), (with* $a_1 = 0$*). To leading (=lowest) order in* $X = X_1 - X_2$*, we have the system of ODEs*

$$\Big(d_{X_s} - \frac{c}{8}\omega_s\Big)\langle\mathbf{1}\rangle = 2\xi_s \frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}} \, ,$$

$$\Big(d_{X_s} - \frac{c}{8}\omega_s\Big)\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}} \cong \xi_s X^{-1}\left[\frac{7c}{80}X^{-1}\langle\mathbf{1}\rangle + \frac{4}{5}\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}}\right]$$

$$\Big(d_{X_s} - \frac{c}{8}\omega_s\Big)\frac{\langle\vartheta'_{X_s}\rangle}{p'_{X_s}} \cong \xi_s \frac{1}{3}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\left[\frac{7c}{80}X^{-1}\langle\mathbf{1}\rangle + \frac{11}{10}\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}}\right] - \xi_s\left[\frac{3}{10}X^{-1}\frac{\langle\vartheta'_{X_s}\rangle}{p'_{X_s}}\right]$$

$$\begin{aligned}\Big(d_{X_s} - \frac{c}{8}\omega_s\Big)\frac{\langle\vartheta''_{X_s}\rangle}{p'_{X_s}} &\cong \xi_s \frac{c}{96}\left[X^{-1}\frac{p^{(4)}_{X_s}}{p'_{X_s}} + \frac{1}{3}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right]^2\right]\langle\mathbf{1}\rangle \\ &\quad + \xi_s\left[\frac{1}{12}\frac{p^{(4)}_{X_s}}{p'_{X_s}}\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}} + \frac{1}{6}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\frac{\langle\vartheta'_{X_s}\rangle}{p'_{X_s}} - \frac{1}{2}X^{-1}\frac{\langle\vartheta''_{X_s}\rangle}{p'_{X_s}}\right] \\ &\quad + 2\xi_s\frac{\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle_r}{[p'_{X_s}]^2} \, ,\end{aligned}$$

$$\begin{aligned}\Big(d_{X_s} - \frac{c}{8}\omega_s\Big)\frac{\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle_r}{[p'_{X_s}]^2} &\cong \xi_s\frac{c}{640}\left[\frac{1}{2}X^{-2}\frac{p^{(4)}_{X_s}}{p'_{X_s}} - \frac{1}{9}X^{-1}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right]^2\right]\langle\mathbf{1}\rangle \\ &\quad + \frac{1}{80}\xi_s\left[2X^{-1}\frac{p^{(4)}_{X_s}}{p'_{X_s}} - \frac{11}{9}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right]^2\right]\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}} \\ &\quad + \frac{1}{20}\xi_s X^{-1}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\frac{\langle\vartheta'_{X_s}\rangle}{p'_{X_s}} - \frac{3}{50}\xi_s X^{-2}\frac{\langle\vartheta''_{X_s}\rangle}{p'_{X_s}} \\ &\quad - \frac{7}{10}\xi_s X^{-1}\frac{\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle_r}{[p'_{X_s}]^2} + O(X_s - X_2) \, .\end{aligned}$$

(Note that with assumption $X_1 = X_2$ *made for the* 5*th equation,* $(p'_{X_s})^{-1}$ *is not defined. We have to pull* p''_{X_s} *out.)*

Note that the first three equations have been shown for arbitrary $g \geq 1$. (The first two have derived from the exact equations (54) and (55). The third is eq. (106).)

Proof. Under the assumptions of the Theorem, the fourth differential equation reads, to leading (=lowest) order in $X = X_1 - X_2$,

$$\left(d_{X_s} - \frac{c}{8}\omega_s\right)\frac{\langle\vartheta''_{X_s}\rangle}{p'_{X_s}} \cong \xi_s\left[\frac{c}{192}\frac{p''_{X_s}}{p'_{X_s}}\frac{p^{(4)}_{X_s}}{p'_{X_s}} + \frac{c}{288}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right]^2\right]\langle\mathbf{1}\rangle$$
$$+ \xi_s\left[\frac{1}{12}\frac{p^{(4)}_{X_s}}{p'_{X_s}}\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}} + \frac{1}{6}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\frac{\langle\vartheta'_{X_s}\rangle}{p'_{X_s}} - \frac{1}{4}\frac{p''_{X_s}}{p'_{X_s}}\frac{\langle\vartheta''_{X_s}\rangle}{p'_{X_s}}\right]$$
$$+ 2\xi_s\frac{\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle_r}{[p'_{X_s}]^2}\,.$$

(For the proof, cf. Appendix E or F.) Furthermore, eqs (98) and (99) apply. The fifth eq. is obtained from eq. (116). □

12.3 Monodromy matrix for $n = 5$

Let $\vec{Y}$ be the fundamental system in the basis that corresponds to the Frobenius expansion in powers of $X = X_1 - X_2$,

$$\vec{y} = \begin{pmatrix} \frac{\langle\mathbf{1}\rangle}{p'_{X_s}} \\ \frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}} \\ \frac{\langle\vartheta'_{X_s}\rangle}{p'_{X_s}} \\ \frac{\langle\vartheta''_{X_s}\rangle}{p'_{X_s}} \\ \frac{\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle_r}{p'_{X_s}} \end{pmatrix} \cong \begin{pmatrix} a(X_1 - X_2)^{u-1} \\ b(X_1 - X_2)^{u-1} \\ c(X_1 - X_2)^{u-1} \\ d(X_1 - X_2)^{u-1} \\ e(X_1 - X_2)^{u-1} \end{pmatrix}, \quad \vec{Y} = (\vec{y}_1, \ldots, \vec{y}_5)$$

For $s = 1$,

$$\frac{d}{dX_1}\vec{y} \cong \left(\sum_{i\neq 1}\frac{c/8}{X_1 - X_i} + \frac{B}{p'_{X_s}}\right)\vec{y}$$

where

$$B = \begin{pmatrix} -\frac{1}{2}p''_{X_1} & 2 & 0 & 0 & 0 \\ \frac{7c}{320}[p''_{X_1}]^2 & \frac{2}{5}p''_{X_1} & 0 & 0 & 0 \\ \frac{7c}{480}p''_{X_1}p^{(3)}_{X_1} & \frac{11}{30}p^{(3)}_{X_s} & -\frac{3}{20}p''_{X_s} & 0 & 0 \\ \frac{c}{192}p''_{X_s}p^{(4)}_{X_s} + \frac{c}{288}[p^{(3)}_{X_s}]^2 & \frac{1}{12}p^{(4)}_{X_s} & \frac{1}{6}p^{(3)}_{X_s} & -\frac{1}{4}p''_{X_s} & 2 \\ 0 & 0 & 0 & \frac{7c}{320}[p''_{X_s}]^2 & \frac{2}{5}p''_{X_s} \end{pmatrix}.$$

Corollary 34. *Using the Frobenius ansatz*

$$
\begin{aligned}
\langle \mathbf{1} \rangle &\cong a(X_s - X_2)^u \\
\frac{\langle \vartheta_{X_s} \rangle}{p'_{X_s}} &\cong b(X_s - X_2)^{u-1} \\
\frac{\langle \vartheta'_{X_s} \rangle}{p'_{X_s}} &\cong c(X_s - X_2)^{u-1} \\
\frac{\langle \vartheta''_{X_s} \rangle}{p'_{X_s}} &\cong d(X_s - X_2)^{u-1} \\
\frac{\langle \vartheta_{X_s} \vartheta''_{X_s} \rangle_r}{[p'_{X_s}]^2} &\cong e(X_s - X_2)^{u-2} \,, \quad u \in \mathbb{R} \,,
\end{aligned}
$$

the system of DEs takes the form (Note that with assumption $X_1 = X_2$ made for the 5th equation, $(p'_{X_s})^{-1}$ is not defined. We have to pull p''_{X_s} out.)

$$
\begin{pmatrix}
\bar{u} & -2 & 0 & 0 & 0 \\
-\frac{7c}{80} & \bar{u} - \frac{9}{5} & 0 & 0 & 0 \\
-\frac{7c}{240}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right]_{-1} & -\frac{11}{30}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right]_{-1} & \bar{u} - \frac{7}{10} & 0 & 0 \\
\frac{c}{96}\left[\frac{p^{(4)}_{X_s}}{p'_{X_s}}\right]_{-1} + \frac{c}{288}\left[\left(\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right)^2\right]_{-1} & \frac{1}{12}\left[\frac{p^{(4)}_{X_s}}{p'_{X_s}}\right]_{-1} & \frac{1}{6}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right]_{-1} & \bar{u} - \frac{1}{2} & -2 \\
\frac{c}{40}\left\{\frac{1}{32}\left[\frac{p^{(4)}_{X_s}}{p'_{X_s}}\right]_{-1} - \frac{1}{144}\left[\left(\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right)^2\right]_{-1}\right\} & \frac{1}{80}\left\{-\frac{11}{9}\left[\left(\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right)^2\right]_{-1} + 2\left[\frac{p^{(4)}_{X_s}}{p'_{X_s}}\right]_{-1}\right\} & \frac{1}{20}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right]_{-1} & \frac{3}{50} & \bar{u} - \frac{13}{10}
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = 0 \,.
$$

Here by $[\,]_{-1}$ we mean to say that we take the coefficient of the order X^{-1} term only. The determinant is

$$
\left\{\left(\bar{u} - \frac{13}{10}\right)\left(\bar{u} - \frac{1}{2}\right) + \frac{3}{25}\right\} \det
\begin{pmatrix}
\bar{u} & -2 & 0 \\
-\frac{7c}{80} & \bar{u} - \frac{9}{5} & 0 \\
-\frac{7c}{240}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right] & -\frac{11}{30}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right]_{-1} & \bar{u} - \frac{7}{10}
\end{pmatrix}
$$

where the first factor vanishes for $\bar{u} = \frac{7}{10}$ and $\bar{u} = \frac{9}{10}$.

The matrix A in the eigenvalue equation $\bar{u}\begin{pmatrix} a \\ b \\ c \\ . \end{pmatrix} = A\begin{pmatrix} a \\ b \\ c \\ . \end{pmatrix}$ is ... and the Jordan normal form reads

$$
\begin{pmatrix}
\frac{7}{10} & 0 & 0 & 0 \\
0 & \frac{7}{10} & 0 & 0 \\
0 & 0 & \frac{11}{10} & * \\
0 & 0 & 0 & *
\end{pmatrix}
$$

Proof. Only the Jordan normal form of A remains to be proved.

$$
\begin{aligned}
0 = \det(A-\lambda) &= \ldots \det \begin{pmatrix} -\lambda & 2 & 0 \\ \frac{7c}{80} & \frac{9}{5}-\lambda & 0 \\ \frac{7c}{240}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right]_{-1} & \frac{11}{30}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right]_{-1} & \frac{7}{10}-\lambda \end{pmatrix} \\
&= \ldots (\frac{7}{10}-\lambda) \det \begin{pmatrix} -\lambda & 2 \\ \frac{7c}{80} & \frac{9}{5}-\lambda \end{pmatrix} \\
&= \ldots (\frac{7}{10}-\lambda) \left\{ \lambda(\lambda - \frac{9}{5}) - \frac{7c}{40} \right\}
\end{aligned}
$$

so eigenvalues are

$$
\lambda_1 = \lambda_2 = \frac{7}{10}\,, \quad \lambda_3 = \frac{11}{10}\,, \quad \lambda_4 = \ldots
$$

To determine the eigenvectors $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ to $\lambda = \frac{7}{10}$, we consider the system

$$
\begin{aligned}
-\frac{7}{10}v_1 + 2v_2 &= 0 \\
\frac{7c}{80}v_1 + \frac{11}{10}v_2 &= 0 \\
\frac{7c}{240}v_1 + \frac{11}{30}v_2 &= 0\,.
\end{aligned}
$$

All three equations are compatible and yield $\vec{v} = \begin{pmatrix} 20 \\ 7 \\ 0 \end{pmatrix}$. Another linearly independent eigenvector is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Thus to the double eigenvalue we have two linearly independent eigenvectors. This proves the claim about the Jordan normal form (the minor 3×3 matrix of A is diagonalisable). □

Remark 35. *There is no logarithmic solution, despite the fact that several eigenvalues have an integer difference (equal to zero).*

Using $-\frac{c}{24} = \frac{11}{60}$, *we find that the eigenvector of* $\lambda_3 = \frac{11}{10}$ *is* $\vec{v} = v_1 \begin{pmatrix} 1 \\ \frac{11}{10} \\ \frac{11}{60}\left[\frac{p^{(3)}_{X_s}}{p'_{X_s}}\right]_{-1} \end{pmatrix}$ *with* $v_1 \in \mathbb{C}^*$.

Remark 36. *We have* $\langle \vartheta_x \rangle = \frac{1}{4}\Theta(x)$. *For* $n = 5$,

$$
\langle \vartheta^{(3)}_{X_s} \rangle = \frac{3!}{4} A_0 = -\frac{9c}{2} \langle \mathbf{1} \rangle
$$

($a_0 = 1$*), so by the differential eq. (54),*

$$
\left(d_{X_s} - \frac{c}{8}\omega_s \right) \langle \vartheta^{(3)}_{X_s} \rangle \cong -9c\xi_s \frac{\langle \vartheta_{X_s} \rangle}{p'_{X_s}}
$$

13 General results

We consider the hyperelliptic Riemann surface

$$\Sigma_g : y^2 = p(x)\,, \quad \deg p = 2g+1, 2g+2\,.$$

with branch points $X_1, X_2, \ldots, X_{2g+1}, X_{2g+2}$ (where X_{2g+2} may be the point at infinity).

13.1 Branch points as primary (twist) fields

Twist fields are a way to make the dependence of N-point functions on the position of the branch points explicit. If $\langle\varphi(x)\ldots\rangle_{X_1,X_2,\ldots,X_{2g+1},X_{2g+2}}$ denotes an N-point function on Σ_g, we can write

$$\langle\varphi(x)\ldots\mathcal{T}(X_1)\mathcal{T}(X_2)\ldots\rangle_{X_3,\ldots,X_{2g+1},X_{2g+2}} := \langle\varphi(x)\ldots\rangle_{X_1,X_2,\ldots,X_{2g+1},X_{2g+2}}\,.$$

As $X_1 \to X_2$, the ramification at X_2 is dissolved, and the surface Σ_g degenerates to a surface of genus $g-1$, with $2g$ branch points $X_3, \ldots, X_{2g+1}, X_{2g+2}$. For $X_1 \approx X_2$, we have an expansion

$$\langle\varphi(x)\ldots\mathcal{T}(X_1)\mathcal{T}(X_2)\ldots\rangle_{X_3,\ldots} = \sum_k (X_1-X_2)^k \langle\varphi(x)\ldots\chi_k^+(X_2)\chi_k^-(X_2)\ldots\rangle_{X_3,\ldots} \tag{117}$$

where χ_k^+ and χ_k^- are primary fields corresponding to the two different sheets. They don't depend on X_1 and so the $N+2$ point functions on the r.h.s. of eq. (117) are defined on the degenerate (genus $g-1$) hyperelliptic surface. The range of k remains to be specified. If $k \geq 0$ and $k=0$ occurs, then

$$\lim_{X_1\to X_2}\langle\mathcal{T}(X_1)\mathcal{T}(X_2)\rangle_{X_3,\ldots} = \lim_{X_1\to X_2}\langle\mathbf{1}\rangle_{X_1,X_2,\ldots} = \langle\chi_0^+(X_2^+)\chi_0^-(X_2^-)\rangle_{X_3,\ldots}\,. \tag{118}$$

Remark 37. 1. *We cannot make a statement about k because we are working with a singular metric which affects the power of* $(X_1 - X_2)$.

2. *We actually have two types of pairs of fields* $\chi^+\otimes\chi^-$*, namely the field* $1_+\otimes 1_-$ *for* $h=0$ *and some other pair* $\varphi_+\otimes\varphi_-$ *for* $h=-\frac{1}{5}$*. Here* $h=-\frac{1}{5}=\bar{h}$*, and somehow* $\frac{11}{10}-\frac{7}{10}=-(h+\bar{h})$.

3. *A problem is that we don't have the global partition function* $\sum_{i\ Fibonacci}|Z_i|^2$*, but the* Z_i *are only defined up to unitary transformation (monodromy). Going around one ramification point gives a factor of* $e^{2\pi i\,\frac{11}{10}}$*, going around the other* $e^{2\pi i\,\frac{7}{10}}$.

Since $\chi^\pm$ is a primary field, it has the OPE with the

$$T(x)\otimes\chi^\pm(X) \mapsto \frac{h_{\chi^\pm}}{(x-X)^2}\,\chi^\pm(X) + \frac{1}{x-X}\,[\chi^\pm]'(X) + \text{reg.}$$

with the Virasoro field. Letting

$$p(x) =: (x-X_1)(x-X_2)\tilde{p}(x)$$

we have by eq. (117)

$$\frac{1}{(x-X_2)^2\tilde{p}}\langle\vartheta(x)\ldots\chi_n^+(X_2^+)\chi_n^-(X_2^-)\ldots\rangle_{X_3,\ldots} = \frac{h_\chi}{(x-X_2)^2}\langle\ldots\chi_n^+(X_2^+)\chi_n^-(X_2^-)\ldots\rangle_{X_3,\ldots}$$
$$+\frac{1}{x-X_2}\langle\ldots(\chi_n^\pm)'(X_2)\,\chi_n^\mp(X_2)\ldots\rangle_{X_3,\ldots} + \text{reg.}$$
$$-\frac{c}{32}\left[\frac{p'}{p}\right]^2\langle\ldots\chi_n^+(X_2^+)\chi_n^-(X_2^-)\ldots\rangle_{X_3,\ldots} \tag{119}$$

Here

$$\left[\frac{p'}{p}\right]^2 = \frac{4}{(x-X_2)^2} + \frac{4}{x-X_2}\frac{\tilde{p}'}{\tilde{p}} + \left[\frac{\tilde{p}'}{\tilde{p}}\right]^2 + O(X_1-X_2).$$

Thus (119) reads

$$\langle\vartheta(x)\ldots\chi_n^+(X_2^+)\chi_n^-(X_2^-)\ldots\rangle_{X_3,\ldots} = \left(h_\chi-\frac{c}{8}\right)\tilde{p}\,\langle\ldots\chi_n^+(X_2^+)\chi_n^-(X_2^-)\ldots\rangle_{X_3,\ldots} + O(x-X_2)\,.$$

By eq. (118), in absence of fields other than ϑ resp. 1, (119) reads,

$$\left(h_\chi-\frac{c}{8}\right)\lim_{X_1\to X_2}\langle\mathbf{1}\rangle_{X_1,X_2,\ldots} = \lim_{X_1\to X_2}\frac{\langle\vartheta(x)\rangle_{X_1,X_2,\ldots}}{\tilde{p}(x)}\,.$$

We may evaluate at $x = X_1$ since $\vartheta(x)\otimes\mathcal{T}(X_1)$ is non-singular ($\langle\vartheta(x)\rangle_{X_1,X_2,\ldots}$ is a polynomial in x), provided X_1 is finite, and use $p'(X_1) = (X_1-X_2)\tilde{p}(X_1)$:

$$\left(h_\chi-\frac{c}{8}\right)\lim_{X_1\to X_2}\langle\mathbf{1}\rangle_{X_1,X_2,\ldots} = \lim_{X_1\to X_2}(X_1-X_2)\frac{\langle\vartheta(X_1)\rangle_{X_1,X_2,\ldots}}{p'(X_1)}\,. \tag{120}$$

(Note that this makes sense since both sides are $\sim (X_1-X_2)^u$ this way.)

We can compare eq. (120) with the ODE (54): If eq. (120) holds before the limit is taken, for $X_1 \approx X_2$ we must have

$$\bar{u} = 2\left(h_{\chi_\pm}-\frac{c}{8}\right)$$

Moreover, in the minimal model with $c = -\frac{22}{5}$, this reproduces

$$\bar{u} = 2\left(h_{\chi_\pm}+\frac{11}{20}\right) = \begin{cases}\frac{11}{10} & h_{\chi_\pm} = 0\\ \frac{7}{10} & h_{\chi_\pm} = -\frac{1}{5}\end{cases}$$

which is correct.

13.2 The number of ODEs in general

Let $g = 1$. We introduce a non-holomorphic (physical) field φ with the OPE

$$T(z)\varphi(u,\bar{u}) \;\mapsto\; \frac{-1/5}{(z-u)^2}\varphi + \frac{1}{z-u}\partial\varphi + \ldots\partial^2\varphi + O(z-u)\,.$$

By comparison,

$$\langle T(z)\varphi(u,\bar{u})\rangle = -\frac{1}{5}\wp(z-u)\langle\varphi(u,\bar{u})\rangle\,.$$

(as there is no periodic function with a first order pole). We have

$$\int_0^1 \wp(z-u|\tau)\,dz = -\frac{\pi^2}{3}E_2(\tau)\,.$$

So when the contour integal is taken along the real period and $\oint dz = 1$ then

$$\partial_\tau \langle \varphi(u,\bar{u})\rangle = \frac{1}{2\pi i}\oint \langle T(z)\varphi(u,\bar{u})\rangle\,dz \sim -\frac{1}{5}E_2\langle \varphi(u,\bar{u})\rangle$$

so $\langle \varphi(u,\bar{u})\rangle \sim \eta^{-2/5}$. This gives the solution for the $h = \bar{h} = -\frac{1}{5}$ conformal block,

$$\langle \varphi(u,\bar{u})\rangle \sim \eta^{-2/5}$$

For every node (singularity) between two tori we can introduce a field 1 or φ. For $g = 2$, there are two tori connected by one node, and we have

node	number of choices	solutions
$\langle \mathbf{1}\rangle$	2	Rogers-Ramanujan functions
$\langle \varphi\rangle$	1	$\eta^{-2/5}$

For $g = 3$ there are three tori (I-III) connected by two nodes. Only the middle torus (II) has two marked points, and inserting a field on either node may give rise to a 2-point function. We obtain

torus I	node 1	torus II	node 2	torus III	number of choices
$\langle \mathbf{1}\rangle$	1	$\langle \mathbf{1}\rangle$	1	$\langle \mathbf{1}\rangle$	2^3
$\langle \mathbf{1}\rangle$	1	$\langle \varphi\rangle$	φ	$\langle \varphi\rangle$	2
$\langle \varphi\rangle$	φ	$\langle \varphi\rangle$	1	$\langle \mathbf{1}\rangle$	2
$\langle \varphi\rangle$	φ	$\langle \varphi\varphi\rangle$	φ	$\langle \varphi\rangle$	3

For $g = 3$, we must have an equation of order 15 for $\langle \mathbf{1}\rangle$.

We need to explain the 3 choices for $\langle \varphi\varphi\rangle$. Consider the torus II with two marked points. It is obtained by sqeezing a genus $g = 2$ surface. On the torus we have a choice between the partition functions only, while on the $g = 2$ surface we have 5. $\langle \varphi\varphi\rangle$ must make up for this difference.

A Proof of Theorem 7

Notation: Let $A \sqcup B$ denote the union of sets A, B with $A \cap B = \emptyset$.

Let $\mathcal{F}$ be the bundle of holomorphic fields. Let $\mathcal{T} \subset \mathcal{F}$ be the subbundle with fiber $\mathbb{C}T$, and let $\mathcal{T}_+ = \varepsilon \oplus \mathcal{T} \subset \mathcal{F}$, where ε is the trivial bundle. For $N \geq 1$, let $I_N := \{x_1, \ldots, x_N\}$, and $\mathcal{P}_N := \mathcal{P}(I_N)$ be the powerset of I_N. For $I \in \mathcal{P}_N$, let Graph(I) be the set of admissible graphs whose vertices are the points of I, and let $\mathrm{Graph}_N = \mathrm{Graph}(I_N)$. For any $N \geq 0$, we consider the map

$$w : \sqcup_{N\geq 0}\mathcal{T}_+^{\boxtimes N} \ \to\ \sqcup_{N\geq 0}\mathcal{T}^{\boxtimes N}$$

defined as follows: For $\varphi \in \mathcal{T}_+^{\boxtimes N}$ over $(x_1, \ldots, x_N) \in U^N \subset \Sigma^N \setminus \Delta_N$ (symmetrised product) with $x \neq x_i \ \forall\, i$,

$$w(1_x \times_s \varphi) = w(\varphi)\,.$$

For $(x_1, \ldots, x_N)$ as above, and $\prod_{i=1}^N T(x_i)p(x_i) \in \Gamma(U^N, \mathcal{T}^{\boxtimes N})$,

$$w\left(\prod_{i=1}^N T(x_i)p(x_i)\right) = \sum_{\Gamma\in\mathrm{Graph}_N} \tilde{w}\left(\Gamma_N, \prod_{i=1}^N T(x_i)p(x_i)\right),$$

where

$$\tilde{w}\left(\Gamma_N, \prod_{i=1}^N T(x_i)p(x_i)\right) = \left(\frac{c}{2}\right)^{\sharp\mathrm{loops}} \prod_{(x_i,x_j)\in\Gamma} \left(\frac{1}{4} f(x_i, x_j)\right) \bigotimes_{k\in A_N\cap E_N{}^c} \vartheta_k \bigotimes_{\ell\in(A_N\cup E_N)^c} T(x_\ell)p_\ell\,.$$

By the theorem about the graphical representation of $\langle T \ldots T\rangle\, p \ldots p$, w is such that

$$\langle\ \rangle = \langle\ \rangle_r \circ w$$

on $\mathcal{T}_+^{\boxtimes N}$. Note that for $I \in \mathcal{P}_N$, $\Gamma_I \in \mathrm{Graph}(I)$,

$$\tilde{w}\left(\Gamma_I \sqcup \Gamma_{I^c}, \prod_{i=1}^N T(x_i)p(x_i)\right) = \tilde{w}\left(\Gamma_I, \prod_{i\in I} T(x_i)p(x_i)\right)\cdot \tilde{w}\left(\Gamma_{I^c}, \prod_{i\in I^c} T(x_i)p(x_i)\right).$$

Here $I^c = I_N \setminus I$.

Since both $\langle\ \rangle$ and $\langle\ \rangle_r$ are linear, We also have for $\varphi, \psi \in \mathcal{T}_+^{\boxtimes N}$,

$$w\left(\sum_{I\in\mathcal{P}_N}\prod_{x\in I}\varphi(x)\prod_{x\in I_N\setminus I}\psi(x)\right) = \sum_{I\in\mathcal{P}_N} w\left(\prod_{x\in I}\varphi(x)\prod_{x\in I_N\setminus I}\psi(x)\right).$$

Now $\vartheta \in \mathcal{T}_+$. For $P = -\frac{c}{32}\frac{[p']^2}{p}.1$,

$$\begin{aligned}
w\left(\prod_{j=1}^N \vartheta_j\right) &= w\left(\prod_{x\in\mathcal{P}_N}(T(x)p_x + P(x))\right)\\
&= w\left(\sum_{I\in\mathcal{P}_N}\prod_{x\in I} T(x)p_x \prod_{x\in I_N\setminus I} P(x)\right)\\
&= \sum_{I\in\mathcal{P}_N}\left(\prod_{x\in I_N\setminus I} P(x)\right)\cdot w\left(\prod_{x\in I} T(x)p_x\right)\\
&= \sum_{I\in\mathcal{P}_N}\left(\prod_{x\in I_N\setminus I} P(x)\right)\sum_{\Gamma\in\mathrm{Graph}(I)} \tilde{w}\left(\Gamma, \prod_{x\in I} T(x)p_x\right)
\end{aligned}$$

Let ess be the projection

$$\mathrm{ess} : \cup_{I\in\mathcal{P}_N}\mathrm{Graph}(I) \;\to\; \mathcal{P}_N$$

which assigns to a graph its (essential support consisting of its) set of non-isolated vertices. For $I \in \mathcal{P}_N$, let $Is(\Gamma) := (A_\Gamma \cup E_\Gamma)^c$, the set of isolated points of $\Gamma \in \mathrm{Graph}(I)$. Let $\Gamma_0(x)$ be the graph consisting of the point x (with no links). Every graph Γ can be written as

$$\Gamma = \Gamma^{\mathrm{red}} \sqcup \left(\cup_{x\in\mathrm{ess}(\Gamma)^c}\Gamma_0(x)\right) = \Gamma^{\mathrm{red}} \sqcup \left(\cup_{x\in\mathrm{Is}(\Gamma)}\Gamma_0(x)\right)$$

(disjoint unions). By the previous computation,

$$\begin{aligned}
w\left(\prod_{j=1}^{N}\vartheta_j\right) &= \sum_{I\in\mathcal{P}_N}\left(\prod_{x\in I_N\setminus I}P(x)\right)\sum_{\Gamma\in\mathrm{Graph}(I)}\tilde{w}\left(\Gamma,\prod_{x\in I}T(x)p_x\right)\\
&= \sum_{I\in\mathcal{P}_N}\left(\prod_{x\in I_N\setminus I}P(x)\right)\sum_{\Gamma\in\mathrm{Graph}(I)}\tilde{w}\left(\Gamma,\prod_{x\in\mathrm{ess}(\Gamma)}T(x)p_x\prod_{x\in\mathrm{Is}(\Gamma)}T(x)p_x\right)\\
&= \sum_{I\in\mathcal{P}_N}\left(\prod_{xI_N\setminus I}P(x)\right)\sum_{\Gamma\in\mathrm{Graph}(I)}\tilde{w}\left(\Gamma^{\mathrm{red}},\prod_{x\in\mathrm{ess}(\Gamma^{\mathrm{red}})}T(x)p_x\right)\left(\prod_{x\in\mathrm{Is}(\Gamma)}T(x)p_x\right)\\
&= \sum_{\tilde{I}\in\mathcal{P}_N}\sum_{\Gamma^{\mathrm{red}}\in\mathrm{Graph}(\tilde{I})}\tilde{w}\left(\Gamma^{\mathrm{red}},\prod_{x\in\mathrm{ess}(\Gamma^{\mathrm{red}})}T(x)p_x\right)\sum_{\substack{I\in\mathcal{P}_N\\ I\supset\tilde{I}}}\left(\prod_{x\in I\setminus\tilde{I}}T(x)p_x\right)\left(\prod_{x\in I_N\setminus I}P(x)\right)\\
&= \sum_{\tilde{I}\in\mathcal{P}_N}\sum_{\Gamma^{\mathrm{red}}\in\mathrm{Graph}(\tilde{I})}\tilde{w}\left(\Gamma^{\mathrm{red}},\prod_{x\in\mathrm{ess}(\Gamma^{\mathrm{red}})}T(x)p_x\right)\prod_{x\in I_N\setminus\tilde{I}}(T(x)p_x+P(x))\\
&= \sum_{\tilde{I}\in\mathcal{P}_N}\sum_{\Gamma^{\mathrm{red}}\in\mathrm{Graph}(\tilde{I})}\tilde{w}\left(\Gamma^{\mathrm{red}},\prod_{x\in\mathrm{ess}(\Gamma^{\mathrm{red}})}T(x)p_x\right)\prod_{x\in I_N\setminus\tilde{I}}\vartheta_x\\
&= \sum_{\tilde{I}\in\mathcal{P}_N}\sum_{\Gamma^{\mathrm{red}}\in\mathrm{Graph}(\tilde{I})}\left(\frac{c}{2}\right)^{\sharp\text{loops of }\Gamma^{\mathrm{red}}}\prod_{(x_i,x_j)\in\Gamma^{\mathrm{red}}}\left(\frac{1}{4}f(x_i,x_j)\right)\bigotimes_{k\in A_{\Gamma^{\mathrm{red}}}\cap E_{\Gamma^{\mathrm{red}}}{}^c}\vartheta_k\prod_{x\in I_N\setminus\tilde{I}}\vartheta_x\\
&= \sum_{\Gamma\in\mathrm{Graph}_N}\left(\frac{c}{2}\right)^{\sharp\text{loops of }\Gamma}\prod_{(x_i,x_j)\in\Gamma}\left(\frac{1}{4}f(x_i,x_j)\right)\bigotimes_{k\in E_\Gamma{}^c}\vartheta_k\,,
\end{aligned}$$

and application of $\langle\;\rangle_r$ yields the claimed formula.

B Sketch of the proof of Lemma 9

By induction. Sketch of the argument: From eq. (23) follows

$$d\langle\vartheta\rangle = p\left\{d\langle T\rangle - \frac{c}{32}\left(\frac{p'}{p}\right)^2 d\langle\mathbf{1}\rangle\right\} + \langle\vartheta\rangle\frac{dp}{p} - \frac{c}{16}\langle\mathbf{1}\rangle pd\left(\frac{p'}{p}\right),$$

where by eq. (42) for $N = 0, 1$,

$$p\left\{d\langle T\rangle - \frac{c}{32}\left(\frac{p'}{p}\right)^2 d\langle \mathbf{1}\rangle\right\} = 2p\sum_{s=1}^{n}\frac{\xi_s}{p'_{X_s}}\left\{\langle\vartheta_{X_s}T\rangle - \frac{c}{32}\left(\frac{p'}{p}\right)^2\langle\vartheta_{X_s}\rangle\right\} + \frac{c}{8}\,\omega\left\{\langle T\rangle - \frac{c}{32}\left(\frac{p'}{p}\right)^2\langle\mathbf{1}\rangle\right\}$$

$$= 2\sum_{s=1}^{n}\frac{\xi_s}{p'_{X_s}}\langle\vartheta_{X_s}\vartheta\rangle + \frac{c}{8}\,\omega\langle\vartheta\rangle\,.$$

For $N = 2$,

$$d\langle\vartheta_1\vartheta_2\rangle = p_1p_2\left\{d\langle T_1T_2\rangle - \frac{c^2}{(32)^2}\frac{[p'_1p'_2]^2}{(p_1p_2)^2}d\langle\mathbf{1}\rangle\right\}$$
$$+\langle\vartheta_1\vartheta_2\rangle\frac{d(p_1p_2)}{p_1p_2} - \frac{c}{16}\langle\vartheta_1\rangle p_1p'_2 d\left(\frac{p'_2}{p_2}\right) - \frac{c}{16}\langle\vartheta_2\rangle p'_1p_2 d\left(\frac{p'_1}{p_1}\right) - \frac{c}{32}\frac{[p'_2]^2}{p_2}d\langle\vartheta_1\rangle - \frac{c}{32}\frac{[p'_1]^2}{p_1}d\langle\vartheta_2\rangle\,.$$

On the other hand, in eq. (42) for $N = 0, 2$

$$p_1p_2\sum_{s=1}^{n}\frac{\xi_s}{p'_{X_s}}\left\{\langle\vartheta_{X_s}T_1T_2\rangle - \frac{c^2}{(32)^2}\left(\frac{p'_1p'_2}{p_1p_2}\right)^2\langle\vartheta_{X_s}\rangle\right\}$$
$$= \sum_{s=1}^{n}\frac{\xi_s}{p'_{X_s}}\langle\vartheta_{X_s}\vartheta_1\vartheta_2\rangle + \frac{c}{32}\frac{[p'_1]^2}{p_1}\sum_{s=1}^{n}\frac{\xi_s}{p'_{X_s}}\langle\vartheta_{X_s}\vartheta_2\rangle + \frac{c}{32}\frac{[p'_2]^2}{p_2}\sum_{s=1}^{n}\frac{\xi_s}{p'_{X_s}}\langle\vartheta_{X_s}\vartheta_1\rangle$$

The last two terms drop out as $-\frac{c}{32}\frac{[p'_2]^2}{p_2}d\langle\vartheta_1\rangle - \frac{c}{32}\frac{[p'_1]^2}{p_1}d\langle\vartheta_2\rangle$ are added.

C Proof of Lemma 10

Application of $\langle\quad\rangle$ to eq. (38) of $\vartheta_x\otimes\vartheta_{X_s}$ yields an identity of states

$$\langle\psi_1\rangle + \langle\psi_2\rangle = 2\langle\vartheta_1\vartheta_2\rangle - 2\{f_{12}\text{-terms}\} + O((x_1 - x_2)^2)\,. \tag{121}$$

where

$$\langle\vartheta_1\vartheta_2\rangle = \langle\vartheta_1^{[1]}\vartheta_2^{[1]}\rangle + y_1y_2\langle\vartheta_1^{[y]}\vartheta_2^{[y]}\rangle + y_1\langle\vartheta_1^{[y]}\vartheta_2^{[1]}\rangle + y_2\langle\vartheta_1^{[1]}\vartheta_2^{[y]}\rangle\,,$$

and

$$\{f_{12}\text{-terms}\} = \frac{c}{32}f_{12}^2\langle\mathbf{1}\rangle + \frac{1}{4}f_{12}\left\{\langle\vartheta_1^{[1]}\rangle + \langle\vartheta_2^{[1]}\rangle\right\} + \frac{1}{4}f_{12}\left\{y_1\langle\vartheta_1^{[y]}\rangle + y_2\langle\vartheta_2^{[y]}\rangle\right\}\,.$$

In the $(2, 5)$ minimal model,

$$\begin{aligned}\langle\psi_2\rangle &= \lim_{x_1\to x_2}\,[\langle\vartheta_1\vartheta_2\rangle]_{\text{no pole}}\\ &= \lim_{x_1\to x_2}\,[\langle\vartheta_1\vartheta_2\rangle - \{f_{12}\text{-terms}\}]\\ &= \lim_{x_1\to x_2}\langle\vartheta_1\vartheta_2\rangle_r\\ &= \langle\vartheta_2\vartheta_2\rangle_r\\ &= \langle\vartheta_2^{[1]}\vartheta_2^{[1]}\rangle_r + p_2\langle\vartheta_2^{[y]}\vartheta_2^{[y]}\rangle_r + 2y_2\langle\vartheta_2^{[y]}\vartheta_2^{[1]}\rangle_r\,.\end{aligned}$$

is known. $\langle\psi\rangle$ has a Galois splitting

$$\langle\psi\rangle = \langle\psi^{[1]}\rangle + y\langle\psi^{[y]}\rangle\,,$$

so $\langle\psi^{[1]}\rangle$ and $\langle\psi^{[y]}\rangle$ are known, where

$$\begin{aligned}\langle\psi_2^{[1]}\rangle &= \langle\vartheta_2^{[1]}\vartheta_2^{[1]}\rangle_r + p_2\langle\vartheta_2^{[y]}\vartheta_2^{[y]}\rangle_r\\ \langle\psi_2^{[y]}\rangle &= 2\langle\vartheta_2^{[1]}\vartheta_2^{[y]}\rangle_r\,.\end{aligned}$$

It follows that also

$$\begin{aligned}\partial_{x_2}\langle\psi_2^{[1]}\rangle &= \partial_{x_2}\langle\vartheta_2^{[1]}\vartheta_2^{[1]}\rangle_r + p_2'\langle\vartheta_2^{[y]}\vartheta_2^{[y]}\rangle_r + p_2\,\partial_{x_2}\langle\vartheta_2^{[y]}\vartheta_2^{[y]}\rangle_r\,,\\ \partial_{x_2}\langle\psi_2^{[y]}\rangle &= 2\partial_{x_2}\langle\vartheta_2^{[1]}\vartheta_2^{[y]}\rangle_r\end{aligned}$$

are known, and thus $\langle(\psi^{[1]})'_{X_s}\rangle$ and $\langle(\psi^{[y]})'_{X_s}\rangle$. To be specific, we go back to eq. (121). Thus

$$\begin{aligned}\partial_1\langle\psi_1^{[1]}\rangle + \partial_2\langle\psi_2^{[1]}\rangle = {}& (\partial_{x_1}+\partial_{x_2})\Big(\langle\vartheta_1^{[1]}\vartheta_2^{[1]}\rangle + y_1y_2\langle\vartheta_1^{[y]}\vartheta_2^{[y]}\rangle\\ &-\left\{\frac{c}{32}f_{12}^2\langle\mathbf{1}\rangle + \frac{1}{4}f_{12}\left\{\langle\vartheta_1^{[1]}\rangle + \langle\vartheta_2^{[1]}\rangle\right\}\right\}\Big) + O(x_1-x_2)\,,\end{aligned}$$

and

$$\begin{aligned}&\partial_{x_2}\langle\psi_2^{[1]}\rangle\\ &= \frac{1}{2}\lim_{x_1\to x_2}\Big[\langle(\vartheta_1^{[1]})'\vartheta_2^{[1]}\rangle + \langle\vartheta_1^{[1]}(\vartheta_2^{[1]})'\rangle + \frac{1}{2}\left\{\frac{p_1'}{p_1}+\frac{p_2'}{p_2}\right\}y_1y_2\langle\vartheta_1^{[y]}\vartheta_2^{[y]}\rangle + y_1y_2\langle(\vartheta_1^{[y]})'\vartheta_2^{[y]}\rangle + y_1y_2\langle\vartheta_1^{[y]}(\vartheta_2^{[y]})'\rangle\\ &\qquad -(\partial_{x_1}+\partial_{x_2})\left\{\frac{c}{32}f_{12}^2\langle\mathbf{1}\rangle + \frac{1}{4}f_{12}\left\{\langle\vartheta_1^{[1]}\rangle+\langle\vartheta_2^{[1]}\rangle\right\}\right\}\Big]\\ &= \frac{1}{2}\lim_{x_1\to x_2}\Big[2\langle\vartheta_1^{[1]}(\vartheta_2^{[1]})'\rangle + p_2'\langle\vartheta_1^{[y]}\vartheta_2^{[y]}\rangle + 2p_2\langle\vartheta_1^{[y]}(\vartheta_2^{[y]})'\rangle - (\partial_{x_1}+\partial_{x_2})\left\{\frac{c}{32}f_{12}^2\langle\mathbf{1}\rangle + \frac{1}{4}f_{12}\left\{\langle\vartheta_1^{[1]}\rangle+\langle\vartheta_2^{[1]}\rangle\right\}\right\}\Big]\,.\end{aligned}$$

We conclude that

$$\boxed{\partial_x|_{X_s}\langle\psi_2^{[1]}\rangle = \langle\vartheta_{X_s}(\vartheta^{[1]})'_{X_s}\rangle_r + \frac{1}{2}p'_{X_s}\langle\vartheta_{X_s}^{[y]}\vartheta_{X_s}^{[y]}\rangle_r}\,.$$

Likewise,

$$\partial_1\langle\psi_1^{[y]}\rangle + \partial_2\langle\psi_2^{[y]}\rangle = \partial_{x_2}\langle\vartheta_1^{[1]}\vartheta_2^{[y]}\rangle + \partial_{x_1}\langle\vartheta_1^{[y]}\vartheta_2^{[1]}\rangle - (\partial_{x_1}+\partial_{x_2})\left\{\frac{1}{4}f_{12}\left\{\langle\vartheta_1^{[y]}\rangle+\langle\vartheta_2^{[y]}\rangle\right\}\right\} + O(x_1-x_2)\,,$$

so

$$\partial_2\langle\psi_2^{[y]}\rangle = \frac{1}{2}\lim_{x_1\to x_2}\left[2\langle\vartheta_1^{[1]}(\vartheta_2^{[y]})'\rangle - (\partial_{x_1}+\partial_{x_2})\left\{\frac{1}{4}f_{12}\left\{\langle\vartheta_1^{[y]}\rangle+\langle\vartheta_2^{[y]}\rangle\right\}\right\}\right]\,,$$

whence

$$\boxed{\partial_x|_{X_s}\langle\psi_x^{[y]}\rangle = \langle\vartheta_{X_s}(\vartheta^{[y]})'_{X_s}\rangle_r}\,,$$

as required.

D Proof of Claim 6

We compute the expressions given by eqs (40) and (41), at $x = X_s$, up to order $(x-X_s)^3$ terms.

1. We first address eq. (40),

$$\left[\langle\vartheta_{X_s}\vartheta_x^{[1]}\rangle\right]_{\text{reg.}} = \left[\frac{c}{32}f_{X_s x}^2\langle\mathbf{1}\rangle + \frac{1}{4}f_{X_s x}\left\{\langle\vartheta_{X_s}\rangle + \langle\vartheta_x^{[1]}\rangle\right\}\right]_{\text{reg.}} + \langle\vartheta_{X_s}\vartheta_x^{[1]}\rangle_r\,,$$

We have

$$\begin{aligned}\frac{2}{p'_{X_s}}\frac{c}{32}f_{xX_s}^2 &= \frac{c}{16}\left(\frac{p'_{X_s}}{(x-X_s)^2} + \frac{p''_{X_s}}{x-X_s} + \frac{1}{4}\frac{[p''_{X_s}]^2}{p'_{X_s}} + \frac{1}{3}p^{(3)}_{X_s}\right)\\ &+ \frac{c}{16}\left(\frac{1}{12}p^{(4)}_{X_s} + \frac{1}{6}\frac{p''_{X_s}}{p'_{X_s}}p^{(3)}_{X_s}\right)(x-X_s)\\ &+ \frac{c}{16}\left(\frac{1}{24}\frac{p''_{X_s}}{p'_{X_s}}p^{(4)}_{X_s} + \frac{1}{60}p^{(5)}(X_s) + \frac{1}{36}\frac{[p^{(3)}_{X_s}]^2}{p'_{X_s}}\right)(x-X_s)^2 + O((x-X_s)^3)\,.\end{aligned}$$

Thus to leading order,

$$\begin{aligned}\frac{2}{p'_{X_s}}\left[\frac{c}{32}f_{xX_s}^2\right]_{\text{reg.}} &\cong \frac{c}{64}\frac{[p''_{X_s}]^2}{p'_{X_s}} + \frac{c}{96}\frac{p''_{X_s}}{p'_{X_s}}p^{(3)}_{X_s}(x-X_s)\\ &+ \frac{c}{192}\left(\frac{1}{2}\frac{p''_{X_s}}{p'_{X_s}}p^{(4)}_{X_s} + \frac{1}{3}\frac{[p^{(3)}_{X_s}]^2}{p'_{X_s}}\right)(x-X_s)^2 + O((x-X_s)^3)\,.\end{aligned} \tag{122}$$

Now we address $\frac{2}{p'_{X_s}}\frac{1}{4}f_{xX_s}\{\vartheta_x + \vartheta_{X_s}\}$. To simplify notations, set

$$\vartheta = \vartheta^{[1]}\,.$$

Now

$$\begin{aligned}&\frac{2}{p'_{X_s}}\frac{1}{4}f_{xX_s}\{\vartheta_x + \vartheta_{X_s}\}\\ &= \frac{\vartheta_{X_s}}{x-X_s} + \frac{1}{2}\frac{p''_{X_s}}{p'_{X_s}}\vartheta_{X_s} + \frac{1}{6}(x-X_s)\frac{p^{(3)}_{X_s}}{p'_{X_s}}\vartheta_{X_s} + \frac{1}{24}(x-X_s)^2\frac{p^{(4)}_{X_s}}{p'_{X_s}}\vartheta_{X_s}\\ &+ \frac{1}{2}\vartheta'_{X_s} + \frac{1}{4}(x-X_s)\frac{p''_{X_s}}{p'_{X_s}}\vartheta'_{X_s} + \frac{1}{12}(x-X_s)^2\frac{p^{(3)}_{X_s}}{p'_{X_s}}\vartheta'_{X_s}\\ &+ \frac{1}{4}(x-X_s)\vartheta''_{X_s} + \frac{1}{8}(x-X_s)^2\frac{p''_{X_s}}{p'_{X_s}}\vartheta''_{X_s}\\ &+ \frac{1}{12}(x-X_s)^2\vartheta^{(3)}_{X_s} + O((x-X_s)^3)\,.\end{aligned} \tag{123}$$

or

$$
\begin{aligned}
\frac{2}{p'_{X_s}}\frac{1}{4}f_{xX_s}\{\vartheta_x+\vartheta_{X_s}\} = {} & \frac{\vartheta_{X_s}}{x-X_s}\\
&+\frac{1}{2}\left(\frac{p''_{X_s}}{p'_{X_s}}\vartheta_{X_s}+\vartheta'_{X_s}\right)\\
&+\frac{1}{6}\left(\frac{p^{(3)}_{X_s}}{p'_{X_s}}\vartheta_{X_s}+\frac{3}{2}\frac{p''_{X_s}}{p'_{X_s}}\vartheta'_{X_s}+\frac{3}{2}\vartheta''_{X_s}\right)(x-X_s)\\
&+\frac{1}{24}\left(\frac{p^{(4)}_{X_s}}{p'_{X_s}}\vartheta_{X_s}+2\frac{p^{(3)}_{X_s}}{p'_{X_s}}\vartheta'_{X_s}+3\frac{p''_{X_s}}{p'_{X_s}}\vartheta''_{X_s}+2\vartheta^{(3)}_{X_s}\right)(x-X_s)^2\\
&+O((x-X_s)^3)\,.
\end{aligned}
$$

Remark 38. *Suppose* $\left[\frac{1}{4}f_{xX_s}\vartheta_x\right]_{reg.}$ *with* $\vartheta=\vartheta^{[1]},\vartheta^{[y]}$ *is known up to terms in* $O((x-X_s)^2)$*. This defines a system*

$$
\begin{aligned}
(p'\vartheta)'_{X_s} = p''_{X_s}\vartheta_{X_s}+p'_{X_s}\vartheta'_{X_s} &= *\\
p^{(3)}_{X_s}\vartheta_{X_s}+\frac{3}{2}\left(p''_{X_s}\vartheta'_{X_s}+p'_{X_s}\vartheta''_{X_s}\right) &= *
\end{aligned}
$$

which is solvable for ϑ_{X_s} *and* ϑ'_{X_s} *as functions of* ϑ''_{X_s} *iff* $S(p)|_{X_s}\neq 0$*. (This follows from eq. (123), using that* $p'_{X_s}\neq 0$*.) For instance, for* $g=1$*,* $\langle\vartheta''\rangle$ *is constant in position.*

It follows that

$$
\begin{aligned}
\frac{2}{p'_{X_s}}\left[\frac{1}{4}f_{xX_s}\{\vartheta_x+\vartheta_{X_s}\}\right]_{\text{reg.}} \cong {} & \frac{1}{2}\frac{p''_{X_s}}{p'_{X_s}}\vartheta_{X_s}\\
&+\left(\frac{1}{6}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\vartheta_{X_s}+\frac{1}{4}\frac{p''_{X_s}}{p'_{X_s}}\vartheta'_{X_s}\right)(x-X_s)\\
&+\frac{1}{4}\left(\frac{1}{6}\frac{p^{(4)}_{X_s}}{p'_{X_s}}\vartheta_{X_s}+\frac{1}{3}\frac{p^{(3)}_{X_s}}{p'_{X_s}}\vartheta'_{X_s}+\frac{1}{2}\frac{p''_{X_s}}{p'_{X_s}}\vartheta''_{X_s}\right)(x-X_s)^2\\
&+O((x-X_s)^3)\,.
\end{aligned}
\tag{124}
$$

Moreover, for the $(2,5)$ minimal model, $\langle\vartheta_{X_s}\vartheta_{X_s}\rangle_r$ is given by eq. (25). Applying $\langle\,\rangle$ to the previous formulae, multiplying by $\frac{p'_{X_s}}{2}$ and summing up yields the claim with $\vartheta=\vartheta^{[1]}$.

2. It remains to consider eq. (41),

$$
\left[\langle\vartheta_{X_s}\vartheta^{[y]}_x\rangle\right]_{\text{reg.}} = \left[\frac{1}{4}f_{X_sx}\langle\vartheta^{[y]}_x\rangle\right]_{\text{reg.}} + \langle\vartheta_{X_s}\vartheta^{[y]}_x\rangle_r\,,
$$

Here $\left[\frac{1}{4}f_{X_sx}\langle\vartheta^{[y]}_x\rangle\right]_{\text{reg.}}$ equals $\frac{1}{2}p'_{X_s}$ times the regular part in eq. (123) for $\vartheta^{[y]}$ in place of ϑ, und $\langle\vartheta_{X_s}\vartheta^{[y]}_{X_s}\rangle_r=\frac{1}{2}\langle\psi^{[y]}_{X_s}\rangle$ is known for the $(2,5)$ minimal model.

E Proof of the fourth differential equation when $n = 5$

We follow the arguments given by eqs (104), (107) and (40). From eq. (109) follows

$$\frac{2}{p'_{X_s}}\frac{\partial^2}{\partial x_2^2}\left[\frac{c}{32}f(X_s,x_2)^2\langle\mathbf{1}\rangle+\frac{1}{4}f(X_s,x_2)\left\{\langle\vartheta_{X_s}\rangle+\langle\vartheta_x\rangle\right\}\right]_{\text{reg.}}$$
$$\cong\frac{c}{96}\langle\mathbf{1}\rangle\left(\frac{1}{2}\frac{p''_{X_s}}{p'_{X_s}}p^{(4)}_{X_s}+\frac{1}{3}\frac{[p^{(3)}_{X_s}]^2}{p'_{X_s}}\right)$$
$$+\frac{1}{12}p^{(4)}_{X_s}\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}}+\frac{1}{6}p^{(3)}_{X_s}\frac{\langle\vartheta'_{X_s}\rangle}{p'_{X_s}}+\frac{1}{4}p''_{X_s}\frac{\langle\vartheta''_{X_s}\rangle}{p'_{X_s}}$$
$$+O(x_2-X_s)\,.$$

From eqs (110) and (112) follows

$$\frac{\partial^2}{\partial x^2}|_{X_s}\langle\vartheta_{X_s}\vartheta_x\rangle_r=\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle_r=\frac{1}{2}\Psi''_{X_s}-\langle\vartheta'_{X_s}\vartheta'_{X_s}\rangle_r\,, \tag{125}$$

where

$$\langle\psi''_x\rangle=\frac{c}{240}[p^{(3)}_x]^2\langle\mathbf{1}\rangle+\frac{c}{480}p''_xp^{(4)}_x\langle\mathbf{1}\rangle-\frac{c}{480}p'_xp^{(5)}_x\langle\mathbf{1}\rangle$$
$$+\frac{1}{5}p^{(4)}_x\langle\vartheta_x\rangle+\frac{3}{10}p^{(3)}_x\langle\vartheta'_x\rangle-\frac{1}{5}p''_x\langle\vartheta''_x\rangle-\frac{1}{2}p'_x\langle\vartheta^{(3)}_x\rangle-\frac{1}{5}p_x\langle\vartheta^{(4)}_x\rangle\,, \tag{126}$$

respectively. (Note that for $g = 2$, $\langle\vartheta^{(4)}(x)\rangle = 0$.) Thus according to eq. (40),

$$\frac{2}{p'_{X_s}}\left[\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle\right]_{\text{reg.}}\cong\left(\frac{7c}{960}\frac{p''_{X_s}}{p'_{X_s}}p^{(4)}_{X_s}+\frac{11c}{1440}\frac{[p^{(3)}_{X_s}]^2}{p'_{X_s}}\right)\langle\mathbf{1}\rangle$$
$$+\frac{17}{60}p^{(4)}_{X_s}\frac{\langle\vartheta_{X_s}\rangle}{p'_{X_s}}+\frac{7}{15}p^{(3)}_{X_s}\frac{\langle\vartheta'_{X_s}\rangle}{p'_{X_s}}+\frac{1}{20}p''_{X_s}\frac{\langle\vartheta''_{X_s}\rangle}{p'_{X_s}}$$
$$-\frac{2}{p'_{X_s}}\langle\vartheta'_{X_s}\vartheta'_{X_s}\rangle_r\,.$$

Multiplication by ξ_s and using eqs (104) yields the claim.

F Alternative proof of the fourth differential equation when $n = 5$

If $X_1 = X_2$, then $f(X_s, x_2) = \tilde{p}_2$ is regular, and

$$\frac{\partial^2}{\partial x_2^2}\left[\frac{c}{32}\tilde{p}_2^2\langle\mathbf{1}\rangle+\frac{1}{4}\tilde{p}_2\left\{\langle\vartheta_{X_s}\rangle+\langle\vartheta_x\rangle\right\}\right]_{\text{reg.}}$$
$$=\frac{c}{16}\left([\tilde{p}'_2]^2+\tilde{p}_2\tilde{p}''_2\right)\langle\mathbf{1}\rangle+\frac{1}{4}\tilde{p}''_2\left\{\langle\vartheta_{X_s}\rangle+\langle\vartheta_x\rangle\right\}+\frac{1}{2}\tilde{p}'_2\langle\vartheta'_x\rangle+\frac{1}{4}\tilde{p}_2\langle\vartheta''_2\rangle\,.$$

In addition,

$$\frac{\partial^2}{\partial x^2}|_{X_s}\langle\vartheta_{X_s}\vartheta_x\rangle_r=\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle_r=\frac{1}{2}\Psi''_{X_s}-\langle\vartheta'_{X_s}\vartheta'_{X_s}\rangle_r\,,$$

where

$$\psi''_{X_s} = \frac{3c}{20}[\tilde{p}'_{X_s}]^2\langle\mathbf{1}\rangle + \frac{c}{20}\tilde{p}_{X_s}\tilde{p}''_{X_s}\langle\mathbf{1}\rangle + \frac{12}{5}\tilde{p}''_{X_s}\langle\vartheta_{X_s}\rangle + \frac{9}{5}\tilde{p}'_{X_s}\langle\vartheta'_{X_s}\rangle - \frac{2}{5}\tilde{p}_{X_s}\langle\vartheta''_{X_s}\rangle\,.$$

So

$$\begin{aligned}\frac{2}{p'_{X_s}}\left[\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle\right]_{\text{reg.}} &= \frac{2}{p'_{X_s}}\Big\{\ \frac{c}{16}\left([\tilde{p}'_{X_s}]^2 + \tilde{p}_{X_s}\tilde{p}''_{X_s}\right)\langle\mathbf{1}\rangle \\ &\qquad + \frac{1}{2}\tilde{p}''_{X_s}\langle\vartheta_{X_s}\rangle + \frac{1}{2}\tilde{p}'_{X_s}\langle\vartheta'_{X_s}\rangle + \frac{1}{4}\tilde{p}_{X_s}\langle\vartheta''_{X_s}\rangle \\ &\qquad + \frac{3c}{40}[\tilde{p}'_{X_s}]^2\langle\mathbf{1}\rangle + \frac{c}{40}\tilde{p}_{X_s}\tilde{p}''_{X_s}\langle\mathbf{1}\rangle \\ &\qquad + \frac{6}{5}\tilde{p}''_{X_s}\langle\vartheta_{X_s}\rangle + \frac{9}{10}\tilde{p}'_{X_s}\langle\vartheta'_{X_s}\rangle - \frac{1}{5}\tilde{p}_{X_s}\langle\vartheta''_{X_s}\rangle \\ &\qquad - \langle\vartheta'_{X_s}\vartheta'_{X_s}\rangle_r\Big\} \\ &= \frac{2}{p'_{X_s}}\Big\{\ \left(\frac{11c}{80}[\tilde{p}'_{X_s}]^2 + \frac{7c}{80}\tilde{p}_{X_s}\tilde{p}''_{X_s}\right)\langle\mathbf{1}\rangle \\ &\qquad + \frac{17}{10}\tilde{p}''_{X_s}\langle\vartheta_{X_s}\rangle + \frac{7}{5}\tilde{p}'_{X_s}\langle\vartheta'_{X_s}\rangle + \frac{1}{20}\tilde{p}_{X_s}\langle\vartheta''_{X_s}\rangle \\ &\qquad - \langle\vartheta'_{X_s}\vartheta'_{X_s}\rangle_r\Big\}\end{aligned}$$

Translate back into untwiddled:

$$\begin{aligned}\frac{2}{p'_{X_s}}\left[\langle\vartheta_{X_s}\vartheta''_{X_s}\rangle\right]_{\text{reg.}} = \frac{2}{p'_{X_s}}\Big\{\ &\left(\frac{11c}{36\cdot 80}[p^{(3)}_{X_s}]^2 + \frac{7c}{12\cdot 160}p''_{X_s}p^{(4)}_{X_s}\right)\langle\mathbf{1}\rangle \\ &+ \frac{17}{120}p^{(4)}_{X_s}\langle\vartheta_{X_s}\rangle + \frac{7}{30}p^{(3)}_{X_s}\langle\vartheta'_{X_s}\rangle + \frac{1}{40}p''_{X_s}\langle\vartheta''_{X_s}\rangle \\ &- \langle\vartheta'_{X_s}\vartheta'_{X_s}\rangle_r\Big\}\end{aligned}$$

G Proof for the second derivative of the 3-point function

By the OPE for ϑ and the fact that $\langle\ \rangle$ is compatible with it,

$$\begin{aligned}
\langle \psi_2 \vartheta_3 \rangle + O(x_1 - x_2) = \langle \vartheta_1 \vartheta_2 \vartheta_3 \rangle - \frac{c}{32} f_{12}^2 \langle \vartheta_3 \rangle - \frac{1}{4} f_{12} \{ \langle \vartheta_1 \vartheta_3 \rangle + \langle \vartheta_2 \vartheta_3 \rangle \} \\
= \frac{c}{32} \left\{ f_{23}^2 \langle \vartheta_1 \rangle + f_{13}^2 \langle \vartheta_2 \rangle \right\} \\
+ \frac{1}{4} f_{12} \{ \langle \vartheta_1 \vartheta_3 \rangle_r - \langle \vartheta_1 \vartheta_3 \rangle + \langle \vartheta_2 \vartheta_3 \rangle_r - \langle \vartheta_2 \vartheta_3 \rangle \} \\
+ \frac{1}{4} f_{23} \{ \langle \vartheta_1 \vartheta_2 \rangle_r + \langle \vartheta_3 \vartheta_1 \rangle_r \} \\
+ \frac{1}{4} f_{13} \{ \langle \vartheta_1 \vartheta_2 \rangle_r + \langle \vartheta_2 \vartheta_3 \rangle_r \} \\
+ \frac{1}{16} f_{12} f_{23} \{ \langle \vartheta_1 \rangle + \langle \vartheta_3 \rangle \} \\
+ \frac{1}{16} f_{23} f_{31} \{ \langle \vartheta_1 \rangle + \langle \vartheta_2 \rangle \} \\
+ \frac{1}{16} f_{12} f_{31} \{ \langle \vartheta_2 \rangle + \langle \vartheta_3 \rangle \} \\
+ \frac{c}{4^3} f_{12} f_{23} f_{31} \langle \mathbf{1} \rangle \\
+ \langle \vartheta_1 \vartheta_2 \vartheta_3 \rangle_r
\end{aligned} \tag{127}$$

On the other hand,

$$\langle \psi_2 \vartheta_3 \rangle = -\frac{c}{480} \left(p_2' p_2^{(3)} - \frac{3}{2} [p_2'']^2 \right) \langle \vartheta_3 \rangle + \frac{1}{5} p_2'' \langle \vartheta_2 \vartheta_3 \rangle - \frac{1}{10} p_2' \langle \vartheta_2' \vartheta_3 \rangle - \frac{1}{5} p_2 \langle \vartheta_2'' \vartheta_3 \rangle$$

We consider and $X_1 = X_2$ ($s = 1$) and

$$p_x = (x - X_2)^2 \tilde{p}_x \,, \quad f_{xX_s} = \tilde{p}_x \,.$$

We denote by $\tilde{p}_{X_s}' = \frac{d}{dx_3}|_{x_3 = X_s} \tilde{p}_3$, etc. Solving eq. (127), evaluated at $x_1 = x_2 = X_s$ ($= X_1 = X_2$), for $\langle \vartheta_{X_s} \vartheta_{X_s} \vartheta_3 \rangle_r$,

$$\begin{aligned}
\langle \vartheta_{X_s} \vartheta_{X_s} \vartheta_3 \rangle_r = & - \frac{c}{4^3} \tilde{p}_{X_s} \tilde{p}_3^2 \langle \mathbf{1} \rangle \\
& + \frac{c}{80} \tilde{p}_{X_s}^2 \langle \vartheta_3 \rangle + \frac{3}{20} \tilde{p}_3^2 \langle \vartheta_{X_s} \rangle - \frac{1}{8} \tilde{p}_{X_s} \tilde{p}_3 \{ \langle \vartheta_{X_s} \rangle + \langle \vartheta_3 \rangle \} \\
& + \frac{9}{10} \tilde{p}_{X_s} \langle \vartheta_{X_s} \vartheta_3 \rangle - \frac{1}{2} \tilde{p}_{X_s} \langle \vartheta_{X_s} \vartheta_3 \rangle_r - \frac{1}{2} \tilde{p}_3 \{ \langle \vartheta_{X_s} \vartheta_{X_s} \rangle_r + \langle \vartheta_3 \vartheta_{X_s} \rangle_r \} \,,
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx_3} \langle \vartheta_{X_s} \vartheta_{X_s} \vartheta_3 \rangle_r = & - \frac{c}{32} \tilde{p}_{X_s} \tilde{p}_3 \tilde{p}_3' \langle \mathbf{1} \rangle \\
& + \frac{c}{80} \tilde{p}_{X_s}^2 \langle \vartheta_3' \rangle + \frac{3}{10} \tilde{p}_3 \tilde{p}_3' \langle \vartheta_{X_s} \rangle - \frac{1}{8} \tilde{p}_{X_s} \tilde{p}_3' \{ \langle \vartheta_{X_s} \rangle + \langle \vartheta_3 \rangle \} - \frac{1}{8} \tilde{p}_{X_s} \tilde{p}_3 \langle \vartheta_3' \rangle \\
& + \frac{9}{10} \tilde{p}_{X_s} \langle \vartheta_{X_s} \vartheta_3' \rangle - \frac{1}{2} \tilde{p}_{X_s} \langle \vartheta_{X_s} \vartheta_3' \rangle_r - \frac{1}{2} \tilde{p}_3' \{ \langle \vartheta_{X_s} \vartheta_{X_s} \rangle_r + \langle \vartheta_3 \vartheta_{X_s} \rangle_r \} - \frac{1}{2} \tilde{p}_3 \langle \vartheta_3' \vartheta_{X_s} \rangle_r \,,
\end{aligned}$$

and

$$
\begin{aligned}
&\frac{d^2}{dx_3^2}\langle\vartheta_{X_s}\vartheta_{X_s}\vartheta_3\rangle_r \\
&= -\frac{c}{32}\tilde{p}_{X_s}[\tilde{p}_3']^2\langle\mathbf{1}\rangle - \frac{c}{32}\tilde{p}_{X_s}\tilde{p}_3\tilde{p}_3''\langle\mathbf{1}\rangle \\
&+ \frac{c}{80}\tilde{p}_{X_s}^2\langle\vartheta_3''\rangle + \frac{3}{10}[\tilde{p}_3']^2\langle\vartheta_{X_s}\rangle + \frac{3}{10}\tilde{p}_3\tilde{p}_3''\langle\vartheta_{X_s}\rangle - \frac{1}{8}\tilde{p}_{X_s}\tilde{p}_3''\{\langle\vartheta_{X_s}\rangle + \langle\vartheta_3\rangle\} - \frac{1}{4}\tilde{p}_{X_s}\tilde{p}_3'\langle\vartheta_3'\rangle - \frac{1}{8}\tilde{p}_{X_s}\tilde{p}_3\langle\vartheta_3''\rangle \\
&+ \frac{9}{10}\tilde{p}_{X_s}\langle\vartheta_{X_s}\vartheta_3''\rangle - \frac{1}{2}\tilde{p}_{X_s}\langle\vartheta_{X_s}\vartheta_3''\rangle_r - \frac{1}{2}\tilde{p}_3''\{\langle\vartheta_{X_s}\vartheta_{X_s}\rangle_r + \langle\vartheta_3\vartheta_{X_s}\rangle_r\} - \tilde{p}_3'\langle\vartheta_3'\vartheta_{X_s}\rangle_r - \frac{1}{2}\tilde{p}_3\langle\vartheta_3''\vartheta_{X_s}\rangle_r\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{d^2}{dx_3^2}|_{x_3=X_s}\langle\vartheta_{X_s}\vartheta_{X_s}\vartheta_3\rangle_r \\
&\qquad= -\frac{c}{32}\tilde{p}_{X_s}[\tilde{p}_{X_s}']^2\langle\mathbf{1}\rangle - \frac{c}{32}\tilde{p}_{X_s}^2\tilde{p}_{X_s}''\langle\mathbf{1}\rangle \\
&\qquad- \frac{9}{50}\tilde{p}_{X_s}^2\langle\vartheta_{X_s}''\rangle + \frac{3}{10}[\tilde{p}_{X_s}']^2\langle\vartheta_{X_s}\rangle + \frac{1}{20}\tilde{p}_{X_s}\tilde{p}_{X_s}''\langle\vartheta_{X_s}\rangle - \frac{1}{4}\tilde{p}_{X_s}\tilde{p}_{X_s}'\langle\vartheta_{X_s}'\rangle \\
&\qquad+ \frac{9}{10}\tilde{p}_{X_s}\langle\vartheta_{X_s}\vartheta_{X_s}''\rangle - \tilde{p}_{X_s}\langle\vartheta_{X_s}\vartheta_{X_s}''\rangle_r - \tilde{p}_{X_s}''\Psi_{X_s} - \tilde{p}_{X_s}'\langle\vartheta_{X_s}'\vartheta_{X_s}\rangle_r\,.
\end{aligned}
$$

Now

$$
\langle\vartheta_{X_s}\vartheta_{X_s}''\rangle = \frac{c}{16}\langle\mathbf{1}\rangle([\tilde{p}_{X_s}']^2 + \tilde{p}_{X_s}\tilde{p}_{X_s}'') + \frac{1}{2}\tilde{p}_{X_s}''\langle\vartheta_{X_s}\rangle + \frac{1}{2}\tilde{p}_{X_s}'\langle\vartheta_{X_s}'\rangle + \frac{1}{4}\tilde{p}_{X_s}\langle\vartheta_{X_s}''\rangle + \langle\vartheta_{X_s}\vartheta_{X_s}''\rangle_r
$$

and

$$
\psi_{X_s} = \frac{c}{80}\tilde{p}_{X_s}^2\langle\mathbf{1}\rangle + \frac{2}{5}\tilde{p}_{X_s}\langle\vartheta_{X_s}\rangle\,,
$$

so

$$
\begin{aligned}
\frac{d^2}{dx_3^2}|_{x_3=X_s}\langle\vartheta_{X_s}\vartheta_{X_s}\vartheta_3\rangle_r &= \frac{c}{80}\left(\tilde{p}_{X_s}^2\tilde{p}_{X_s}'' + 2\tilde{p}_{X_s}[\tilde{p}_{X_s}']^2\right)\langle\mathbf{1}\rangle \\
&+ \frac{9}{200}\tilde{p}_{X_s}^2\langle\vartheta_{X_s}''\rangle + \frac{1}{10}\left(\tilde{p}_{X_s}\tilde{p}_{X_s}'' + 3[\tilde{p}_{X_s}']^2\right)\langle\vartheta_{X_s}\rangle + \frac{1}{5}\tilde{p}_{X_s}\tilde{p}_{X_s}'\langle\vartheta_{X_s}'\rangle \\
&- \frac{1}{10}\tilde{p}_{X_s}\langle\vartheta_{X_s}\vartheta_{X_s}''\rangle_r - \tilde{p}_{X_s}'\langle\vartheta_{X_s}'\vartheta_{X_s}\rangle_r\,.
\end{aligned}
$$

H Proof of Claim 12

Let

$$
\begin{aligned}
X_0 &= \xi_0\left(1 + a_1\varepsilon^4\left(\xi_0^2 - 2\tilde{a}_1\right) + \left(a_2\xi_0^3 - 5\tilde{a}_1a_2\xi_0 - 3a_2\tilde{a}_2\right)\varepsilon^6 + O(\varepsilon^8)\right)^{-1} \\
&= \xi_0\sum_{k=0}^{\infty}\left(-a_1\varepsilon^4\left(\xi_0^2 - 2\tilde{a}_1\right) - \left(a_2\xi_0^3 - 5\tilde{a}_1a_2\xi_0 - 3a_2\tilde{a}_2\right)\varepsilon^6 + O(\varepsilon^8)\right)^k\,,
\end{aligned}
$$

and so for $k = 0, 1, 2$,

$$
X_k = \xi_k\left(1 - a_1\varepsilon^4\left(\xi_k^2 - 2\tilde{a}_1\right) - a_2\varepsilon^6\left(\xi_k^3 - 5\tilde{a}_1\xi_k - 3\tilde{a}_2\right) + O(\varepsilon^8)\right)\,.
$$

We have

$$X_1 - X_2 = (\xi_1-\xi_2) - a_1\varepsilon^4\left(\xi_1^3-\xi_2^3-2\tilde a_1(\xi_1-\xi_2)\right) - \left(a_2(\xi_1^4-\xi_2^4) - 5\tilde a_1 a_2(\xi_1^2-\xi_2^2) - 3a_2\tilde a_2(\xi_1-\xi_2)\right)\varepsilon^6 + O(\varepsilon^8)$$
$$X_1 - X_0 = (\xi_1-\xi_0) - a_1\varepsilon^4\left(\xi_1^3-\xi_0^3-2\tilde a_1(\xi_1-\xi_0)\right) - \left(a_2(\xi_1^4-\xi_0^4) - 5\tilde a_1 a_2(\xi_1^2-\xi_0^2) - 3a_2\tilde a_2(\xi_1-\xi_0)\right)\varepsilon^6 + O(\varepsilon^8)\,.$$

So

$$\begin{aligned}
\frac{X_1-X_2}{X_1-X_0} &= (\xi_1-\xi_2)\Big(1 - a_1\varepsilon^4\left(\frac{\xi_1^3-\xi_2^3}{\xi_1-\xi_2}-2\tilde a_1\right) - a_2\varepsilon^6\left(\frac{\xi_1^4-\xi_2^4}{\xi_1-\xi_2}-5\tilde a_1\frac{\xi_1^2-\xi_2^2}{\xi_1-\xi_2}-3\tilde a_2\right)+O(\varepsilon^8)\Big)\times\\
&\quad\times\frac{1}{\xi_1-\xi_0}\sum_{k=0}^{\infty}\Big(a_1\varepsilon^4\left(\frac{\xi_1^3-\xi_0^3}{\xi_1-\xi_0}-2\tilde a_1\right)+a_2\varepsilon^6\left(\frac{\xi_1^4-\xi_0^4}{\xi_1-\xi_0}-5\tilde a_1\frac{\xi_1^2-\xi_0^2}{\xi_1-\xi_0}-3\tilde a_2\right)+O(\varepsilon^8)\Big)^k\\
&= \frac{\xi_1-\xi_2}{\xi_1-\xi_0}\Big(1 - a_1\varepsilon^4\left(\frac{\xi_1^3-\xi_2^3}{\xi_1-\xi_2}-2\tilde a_1\right) - a_2\varepsilon^6\left(\frac{\xi_1^4-\xi_2^4}{\xi_1-\xi_2}-5\tilde a_1\frac{\xi_1^2-\xi_2^2}{\xi_1-\xi_2}-3\tilde a_2\right)+O(\varepsilon^8)\Big)\times\\
&\quad\times\Big(1 + a_1\varepsilon^4\left(\frac{\xi_1^3-\xi_0^3}{\xi_1-\xi_0}-2\tilde a_1\right)+a_2\varepsilon^6\left(\frac{\xi_1^4-\xi_0^4}{\xi_1-\xi_0}-5\tilde a_1\frac{\xi_1^2-\xi_0^2}{\xi_1-\xi_0}-3\tilde a_2\right)+O(\varepsilon^8)\Big)\\
&= \frac{\xi_1-\xi_2}{\xi_1-\xi_0}\Big(1 + a_1\varepsilon^4\Big(\frac{\xi_1^3-\xi_0^3}{\xi_1-\xi_0}-\frac{\xi_1^3-\xi_2^3}{\xi_1-\xi_2}\Big)+a_2\varepsilon^6\Big(\frac{\xi_1^4-\xi_0^4}{\xi_1-\xi_0}-\frac{\xi_1^4-\xi_2^4}{\xi_1-\xi_2}-5\tilde a_1(\xi_0-\xi_2)\Big)+O(\varepsilon^8)\Big)\,.
\end{aligned}$$

Here

$$\begin{aligned}
\frac{\xi_1^3-\xi_0^3}{\xi_1-\xi_0}-\frac{\xi_1^3-\xi_2^3}{\xi_1-\xi_2} &= \xi_1^2+\xi_0\xi_1+\xi_0^2-(\xi_1^2+\xi_1\xi_2+\xi_2^2)\\
&= \xi_0\xi_1+\xi_0^2-\xi_1\xi_2-\xi_2^2\\
&= (\xi_0-\xi_2)(\xi_0+\xi_1+\xi_2) = 0\,,
\end{aligned}$$

by eq. (87) (nicely, this vanishing also works for the other combinations of $\frac{X_i-X_j}{X_i-X_k}$.) We also note that $\xi_0-\xi_2=\frac14\vartheta_4^4$. Moreover,

$$\begin{aligned}
\frac{x-X_0}{x-X_2} &= (x-X_0)\Big(x-\xi_2\Big(1-a_1\varepsilon^4\left(\xi_2^2-2\tilde a_1\right)-a_2\varepsilon^6\left(\xi_2^3-5\tilde a_1\xi_2-3\tilde a_2\right)+O(\varepsilon^8)\Big)\Big)^{-1}\\
&= \Big(1-\frac{\xi_0}{x}\left(1-a_1\varepsilon^4\left(\xi_0^2-2\tilde a_1\right)+O(\varepsilon^6)\right)\Big)\times\\
&\quad\times\Big(1+\sum_{k\ge1}\left(\frac{\xi_2}{x}\right)^k\left(1-ka_1\varepsilon^4\left(\xi_2^2-2\tilde a_1\right)+O(\varepsilon^6)\right)\Big)\\
&= 1-\frac{\xi_0}{x}\left(1-a_1\varepsilon^4\left(\xi_0^2-2\tilde a_1\right)+O(\varepsilon^6)\right)+\sum_{k\ge1}\left(\frac{\xi_2}{x}\right)^k\left(1-ka_1\varepsilon^4\left(\xi_2^2-2\tilde a_1\right)+O(\varepsilon^6)\right)\\
&\quad-\frac{\xi_0}{x}\left(1-a_1\varepsilon^4\left(\xi_0^2-2\tilde a_1\right)+O(\varepsilon^6)\right)\sum_{k\ge1}\left(\frac{\xi_2}{x}\right)^k\left(1-ka_1\varepsilon^4\left(\xi_2^2-2\tilde a_1\right)+O(\varepsilon^6)\right)\,.
\end{aligned}$$

So

$$\begin{aligned}
\frac{(\varepsilon^2\hat X_k)^{-1}-X_0}{(\varepsilon^2\hat X_k)^{-1}-X_2} &= 1-\xi_0\varepsilon^2\hat X_k\left(1-a_1\varepsilon^4\left(\xi_0^2-2\tilde a_1\right)+O(\varepsilon^6)\right)+\sum_{m\ge1}\left(\xi_2\varepsilon^2\hat X_k\right)^m\left(1-ma_1\varepsilon^4\left(\xi_2^2-2\tilde a_1\right)+O(\varepsilon^6)\right)\\
&\quad-\xi_0\varepsilon^2\hat X_k\left(1-a_1\varepsilon^4\left(\xi_0^2-2\tilde a_1\right)+O(\varepsilon^6)\right)\sum_{m\ge1}\left(\xi_2\varepsilon^2\hat X_k\right)^m\left(1-ma_1\varepsilon^4\left(\xi_2^2-2\tilde a_1\right)+O(\varepsilon^6)\right)\\
&= 1+\varepsilon^2\hat X_k(\xi_2-\xi_0)+\varepsilon^4\hat X_k^2\xi_2(\xi_2-\xi_0)+O(\varepsilon^6)\,.
\end{aligned}$$

So the linear fractional transformation

$$x \mapsto f(x) = \frac{X_1 - X_2}{X_1 - X_0}\frac{x - X_0}{x - X_2}$$

maps X_0, X_1, X_2 to $0, 1, \infty$, respectively, and X_{k+3} $(k = 0, 1, 2)$ to

$$\begin{aligned} f\left(\frac{1}{\varepsilon^2 \hat{X}_k}\right) &= \frac{\xi_1 - \xi_2}{\xi_1 - \xi_0}\Big(1 + O(\varepsilon^6)\Big)\Big(1 + \varepsilon^2 \hat{X}_k\Big(\xi_2 - \xi_0\Big) + \xi_2^2 \varepsilon^4 \hat{X}_k^2 + \varepsilon^4 \hat{X}_k^2 \xi_2(\xi_2 - \xi_0) + O(\varepsilon^6)\Big)\Big) \\ &= \frac{\vartheta_3^4}{\vartheta_2^4}\Big(1 + O(\varepsilon^6)\Big)\Big(1 - \frac{\vartheta_4^4}{4}\varepsilon^2 \hat{X}_k + \varepsilon^4 \xi_2^2 \hat{X}_k^2 + O(\varepsilon^6)\Big) . \end{aligned}$$

On the other hand, the linear fractional transformation

$$x \mapsto f(x) = \frac{\xi_1 - \xi_2}{\xi_1 - \xi_0}\frac{x - \xi_0}{x - \xi_2}$$

maps ξ_0, ξ_1, ξ_2 to $0, 1, \infty$, respectively, and maps ξ_{k+3} $(k = 0, 1, 2)$ to

$$\begin{aligned} f\left(\frac{1}{\varepsilon^2 \hat{\xi}_k}\right) &= \frac{\xi_1 - \xi_2}{\xi_1 - \xi_0}(1 - \varepsilon^2 \xi_0 \hat{\xi}_k)\sum_{m=0}^{\infty}(\varepsilon^2 \xi_2 \hat{\xi}_k)^m \\ &= \frac{\xi_1 - \xi_2}{\xi_1 - \xi_0}\Big(1 - \varepsilon^2 \hat{\xi}_k(\xi_0 - \xi_2) - \varepsilon^4 \hat{\xi}_k^2 \xi_2(\xi_0 - \xi_2)) + O(\varepsilon^6) . \end{aligned}$$

Here $\frac{\xi_1 - \xi_2}{\xi_1 - \xi_0} = \frac{\vartheta_3^4}{\vartheta_2^4}$ and $\xi_0 - \xi_2 = \frac{1}{4}\vartheta_4^4$.

I Proof of Claim 16

We have

$$X^{3,j,3,\ell}_{2,j,2,\ell} - X^{3,j',3,\ell'}_{2,j',2,\ell'} = \frac{\vartheta^4_{3,\Omega_{11}}}{\vartheta^4_{2,\Omega_{11}}}\Big(R^{3,j,3,\ell}_{2,j,2,\ell} - R^{3,j',3,\ell'}_{2,j',2,\ell'}\Big) ,$$

where either $j = j' = 3$ (the case $X_3 - X_5$) or $\ell = j' = 2$ (the case $X_3 - X_4$) or $\ell = \ell' = 4$ (the case $X_4 - X_5$). The case $X_3 - X_4$: Here $\ell = j' = 2$, and

$$\begin{aligned} R^{3,3,3,2}_{2,3,2,2} - R^{3,2,3,4}_{2,2,2,4} &= 4\nu^2 \,(R^{(1)}_{3,3} - R^{(1)}_{2,3} - R^{(1)}_{3,4} + R^{(1)}_{2,4}) \\ &+ 4\nu^4 \left(4R^{(1)}_{3,3}R^{(1)}_{3,2} + 4R^{(1)}_{2,3}R^{(1)}_{2,2} + \left[R^{(1)}_{3,3}\right]^2 + 3\left[R^{(1)}_{2,3}\right]^2\right) - 4\nu^4 \left(4R^{(1)}_{3,2}R^{(1)}_{3,4} + 4R^{(1)}_{2,2}R^{(1)}_{2,4} + \left[R^{(1)}_{3,2}\right]^2 + 3\left[R^{(1)}_{2,2}\right]^2\right) \\ &- 16\nu^4 \left(R^{(1)}_{3,3}\left(R^{(1)}_{2,3} + R^{(1)}_{2,2}\right) + R^{(1)}_{3,2}R^{(1)}_{2,3}\right) + 16\nu^4 \left(R^{(1)}_{3,2}R^{(1)}_{2,4} + R^{(1)}_{3,4}\left(R^{(1)}_{2,2} + R^{(1)}_{2,4}\right)\right) \\ &+ \frac{4}{3}\nu^4 \left(R^{(2)}_{3,3} - R^{(2)}_{2,3} - R^{(2)}_{3,4} + R^{(2)}_{2,4}\right) \\ &+ O(\nu^6) . \end{aligned}$$

The case $X_3 - X_5$: Here $j = j' = 3$, and

$$\begin{aligned} R^{3,3,3,2}_{2,3,2,2} - R^{3,3,3,4}_{2,3,2,4} &= 4\nu^2 \,(R^{(1)}_{3,2} - R^{(1)}_{3,4} + R^{(1)}_{2,4} - R^{(1)}_{2,2}) \\ &+ 4\nu^4 \left(4R^{(1)}_{3,3}R^{(1)}_{3,2} + 4R^{(1)}_{2,3}R^{(1)}_{2,2} + \left[R^{(1)}_{3,2}\right]^2 + 3\left[R^{(1)}_{2,2}\right]^2\right) - 4\nu^4 \left(4R^{(1)}_{3,3}R^{(1)}_{3,4} + 4R^{(1)}_{2,3}R^{(1)}_{2,4} + \left[R^{(1)}_{3,4}\right]^2 + 3\left[R^{(1)}_{2,4}\right]^2\right) \\ &- 16\nu^4 \left(R^{(1)}_{3,3}R^{(1)}_{2,2} + R^{(1)}_{3,2}R^{(1)}_{2,3}\right) + 16\nu^4 \left(R^{(1)}_{3,3}R^{(1)}_{2,4} + R^{(1)}_{3,4}R^{(1)}_{2,3}\right) \\ &+ \frac{4}{3}\nu^4 \left(R^{(2)}_{3,2} - R^{(2)}_{2,2}\right) - \frac{4}{3}\nu^4 \left(R^{(2)}_{3,4} - R^{(2)}_{2,4}\right) \\ &+ O(\nu^6) . \end{aligned}$$

Now we have

$$
\begin{aligned}
\frac{X_3 - X_4}{X_3 - X_5} = \frac{R^{3,3,3,2}_{2,3,2,2} - R^{3,2,3,4}_{2,2,2,4}}{R^{3,3,3,2}_{2,3,2,2} - R^{3,3,3,4}_{2,3,2,4}} &= \frac{R^{(1)}_{3,3} - R^{(1)}_{2,3} - R^{(1)}_{3,4} + R^{(1)}_{2,4} + O(\nu^2)}{R^{(1)}_{3,2} - R^{(1)}_{2,2} - R^{(1)}_{3,4} + R^{(1)}_{2,4}} \left(1 + O(\nu^2)\right) \\
&= \frac{\left(\frac{\vartheta'_{3,\Omega_{11}}}{\vartheta_{3,\Omega_{11}}} - \frac{\vartheta'_{2,\Omega_{11}}}{\vartheta_{2,\Omega_{11}}}\right)\left(\frac{\vartheta'_{3,\Omega_{22}}}{\vartheta_{3,\Omega_{22}}} - \frac{\vartheta'_{4,\Omega_{22}}}{\vartheta_{4,\Omega_{22}}}\right)}{\left(\frac{\vartheta'_{3,\Omega_{11}}}{\vartheta_{3,\Omega_{11}}} - \frac{\vartheta'_{2,\Omega_{11}}}{\vartheta_{2,\Omega_{11}}}\right)\left(\frac{\vartheta'_{2,\Omega_{22}}}{\vartheta_{2,\Omega_{22}}} - \frac{\vartheta'_{4,\Omega_{22}}}{\vartheta_{4,\Omega_{22}}}\right)} \left(1 + O(\nu^2)\right) \\
&= \frac{\frac{\vartheta'_{3,\Omega_{22}}}{\vartheta_{3,\Omega_{22}}} - \frac{\vartheta'_{4,\Omega_{22}}}{\vartheta_{4,\Omega_{22}}}}{\frac{\vartheta'_{2,\Omega_{22}}}{\vartheta_{2,\Omega_{22}}} - \frac{\vartheta'_{4,\Omega_{22}}}{\vartheta_{4,\Omega_{22}}}} (1 + O(\nu^2)) \\
&= \frac{\vartheta^4_{2,\Omega_{22}}}{\vartheta^4_{3,\Omega_{22}}} (1 + O(\nu^2))
\end{aligned}
$$

Addendum: We have

$$
\frac{X_3 - X_4}{X_3 - X_5} = \frac{\vartheta^4_{2,\Omega_{22}}}{\vartheta^4_{3,\Omega_{22}}} (1 + O(\nu^2)) = (16\rho_2^{1/2} + O(\rho_2))(1 + O(\nu^2)) \,,
$$

so when ρ_2 is small, so is the distance between X_3 and X_4.

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